Differential game approach to pricing and advertising decisions

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Abstract

This study proposes a model to make concurrent decisions on dynamic pricing and advertising to maximise firms' profitability over an infinite time horizon in a duopoly market. To this end, the Nerlove-Arrow pricing and advertising model is designed in the presence of shifting costs in a dynamic duopolistic competition as a differential game. The Nash equilibrium solution is defined based upon a set of Hamilton–Jacobi–Bellman. Four scenarios are applied for economic interpretations and the efficacy of the model.

Keywords: Dynamic pricing; Dynamic advertising; Duopoly market; Shift costs; Differential game.

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1. Introduction

Marketing activities, namely pricing and advertising decisions often interact over time, having an impact on demand and profit. That is why many researchers have studied both pricing and advertising policies together. Considering a competitive setting where one player's decision can have a direct influence on other players profit, the role of marketing activities becomes even more significant. Moreover, due to technological enhancements and globalisation, pricing and advertising policies are altering rapidly and continuously; therefore, the respective decisions need to be made in a dynamic environment rather than a static one which raises the complexity of decision-making problems [30]. Furthermore, the evolution of advanced technologies in communication, transportation, and delivery systems tends to be a reason for the wane of the long-term relationship between buyers and suppliers. As a result, the possibility to change suppliers has been raised. At the same time, customers should endure the cost of shifting their supplier, the so-called shift cost (SC) [20]. McFarlan (1984) concluded that the firms' tendency to increase their SC has been lifted as a practical solution to hold their customers captive and prevent them from immigrating to rival companies [30].

Accordingly, active firms in today's competitive environment need to identify optimal price and advertising policies to maximise their profits. In addition, it should be noticed that decision-making on pricing and advertising is reliant on the number of firms in the market [23]. The two firms constitute a duopoly market in which the presence of SC can be a pivotal factor to make fortune by existing customers. In this regard, some scholars focused on pricing and advertising decisions apropos of two-echelon distribution channels and supply chains using statistics or differential games (see, e.g., [3]; [27]; [31]). Notably, differential games have frequently applied to dynamic situations in duopoly markets ([30]).

This study makes an attempt to develop a model of dynamic pricing and advertising decisions to maximise the profit of firms for an infinite time horizon in duopoly markets when considering the shifting costs. In the view of shifting costs coupled with an infinite time horizon in duopoly markets, the literature lacks research on how to make dynamic and optimal price and advertising decisions. The existent models in the literature are not successfully implemented in real-world problems due to the fact that many various unsuitable presuppositions are made to simplify the complexity of models. Our model, however, allows a closed-loop solution and provides equilibrium trajectories for price and advertising decisions, followed by numerical scenarios in Iran's Online Cellular data providers with two main players. Although our study focuses only on Cellular data providers, the main findings can be generalized to other industries, as long as the empirical settings are similar.

We use the Nerlove and Arrow (N-A) model in this research because this model has received a lot of attention in the literature over the past few years ([26]; [13]; [32]; [16]). Moreover, this well-known N-A model aims to study the effect of advertising policy on demand over time as well as to consider advertising as a valuable investment for enterprises. Thus, this research's focus moves towards the N-A model with respect to the costs of shifting and dynamic competition in the realm of dynamic pricing and advertisement.

The rest of this paper is outlined as follows. Section 2 reviews the related literature and delineates the research framework. Section 3 provides preliminaries for differential games for determining the optimal price and advertising strategies. Moreover, the proposed model and its properties are discussed in detail. Section 4 illustrates the proposed model by applying it to the four numerical scenarios. Finally, Section 5 gives a summary of the results and possible future research directions.

2. Literature Review and research framework

Profit maximization is a fundamental aspect of revenue management which is in a close relationship with dynamic pricing. The subject has been summarized in the book by [34]. The relating literature on dynamic pricing has been comprehensively reviewed by [9].

The literature mostly considers analysing pure dynamic advertising models in an oligopoly setting including [35], [17] and [18]. Moreover, [26] have classified the dynamic advertising models into six categories; (1) Vidale and Wolfe models [36], (2) Nerlove and Arrow models [13], (3) Lanchester models [29], (4) dynamic advertising and competition models [24], (5) diffusion models [44], and (6) experimental studies.

While joint dynamic pricing and advertising models are extensive, we shall refrain from reviewing the whole literature and focus on models driven from classic dynamic advertising models. (See Table 1). The key aim of dynamic pricing and advertising strategies is to derive an approach for optimal joint pricing and advertising decision variables. Each paper is characterized in terms of seven features shown in Table 1.

Table 1. Comparing recent studies on dynamic pricing and advertising

Study	Variable	Market	Time	Demand	Supply chain	Basis model	Modelling
[13]	P, A	M	*F*	L	Ma, C	NA	DP
[12]	A	O	I	D	Ma, R, C	S	CL, SE
[14]	A	O	F	D	Ma, C	La	CL

Study	Variable	Market	Time	Demand	Supply chain	Basis model	Modelling
[45]	P, A	M	F	L	Ma, R, C	NA	SE
[8]	P, A	M	F	L	Ma, R, C	NA	SE
[43]	P, A, I	M	F	L	Ma, C	NA	DP
[25]	P, A	O	I	E	Ma, C	S	CL
[21]	P, A	M	F	L	Ma, C	NA	DP
[39]	P, A	M	F	E	Ma, C	MR	CL

P=Price, A= Advertisement, I= Inventory
M= Monopoly; O=Oligopoly
F= Finite- time horizon, I= Infinite- time horizon;
L= Linear demand function, D= Dynamic demand function, E= Exponential demand function
Ma= Manufacturer, R=Retailer, C=Customer
NA= Nerlove and Arrow model, S= Sethi model, La= Lanchester model, MR=Macdonald Rasmussen model
CL= Closed-loop approach, SE= Stackelberg equilibrium approach, DP= Dynamic programming approach

Table 2 compares recent research studies on shifting cost and dynamic competition in duopoly markets in terms of time system, time horizon, and utility function.

Table 2. Comparing recent studies in shifting cost and dynamic competition

Study	Research objective	Time system	Time	Utility function
[38]	Determining the equilibrium price of retailers in the presence of SC in the power industry	Ct	F	Stochastic and distributed during a hoteling line
[28]	Determining the behavioural pricing strategies for heterogeneous customers in the presence of SC	Dt	F	Uniform distribution
[19]	Modelling price competition in the presence of SC	Ct	I	Uniform distribution
[1]	Studying the short-term and long-term effect of SC with the price change	Dt	I	The semi-linear function of SC and price
[37]	Offering a traceable model of dynamic competition with a diverse spectrum of SCs	Dt	I	Uniform distribution for old customers

Dt= Discrete time system, Ct= Continuous time system

According to the above discussions, there are two active fields of dynamic pricing and advertising. Hence, the aim of this paper is to explore these fields simultaneously and discuss how shifting costs and dynamic competition create value for organizations. Moreover, an alternative solving approach under competition conditions and marketing variables is implemented. Although [30] attempted to fill the gap by employing the feedback games approach and Sethi model, to the best of our knowledge, no study has been treated dynamic pricing and advertising problems by shifting costs via the N-A model and measuring the feedback equilibrium. The current study aims to look into these lacks in the continuous interval and infinite horizon time.

3. Proposed model

In this section, we first provide preliminaries for differential games including both closed-loop and open-loop and then review the Nerlove and Arrow dynamic model. Afterwards, the problem and relevant notations are defined to formulate the mathematical model and determine equilibrium paths for price and advertising.

3.1. Differential games and equilibriums

A differential game is a dynamic game where players take actions consecutively in response to the other player's actions over time. Therefore, time plays a crucial role in a dynamic game. More importantly, differential games can be regarded as a generalisation of optimal control theory and dynamic programming in which more than one player is involved in the game. However, the differential game is more complicated than optimal control problems [42]. Let us here review some definitions.

Definition 1 (Nash differential games). Assume that there are *N* players ($N \geq 2$), $a^i \epsilon A$; $i = 1,2,\dots,N$ denotes control variables of the players, and A^i is the set of controls that can be selected by the i^{th} player. Given that $\dot{x}$ is the vector-valued differential equation, the state equation is defined as follows.

$$\dot{x} = f(x(t), a^1, a^2, \dots, a^N, t)$$

where *t* represents time, which can be either discrete ($t \in Z$) or continuous ($t \in R$), $x(t)$ is the state variable at a time (*t*). Assume that $g^i(x(t), u(t), t)$ is the running function for player i where u is the input variable of a dynamic system. Moreover, the following value function shows that each player is desired to maximise his payoff:

$$max\left(y^i(.)\right) = \int_0^T g^i(x(t), a^1, a^2, \dots, a^N, t)dt + r^i(x(T))$$

where $y^i(.)$ presents the payoff of player i, and r^i indicates the terminal function for the final period. The summation of the optimal trajectory of the Nash equilibrium for each player is denoted by $\{a^{1*}, a^{2*}, \dots, a^{N*}\}$. The existence condition of the Nash equilibrium in zero-sum differential games is the difference between an open-loop game and a closed-loop game [5].

Definition 2 (Open-loop game). All the equilibrium paths of an open-loop problem are referred to as an open-loop Nash equilibrium. The equilibrium paths are a function of time, i.e., $a^{i*} = a^{i*}(t)$. The following Hamilton – Jacobs – Bellman (HJB) equation is used to check the sufficient condition for the existence of a Nash equilibrium solution:

$H^i = g^i + \lambda^i f$

where $\dot{\lambda}^i = -H_x^i,\ i = 1,2,\dots,N$ and $\lambda^i(T) = r_x^i(x(T))$. Let $a^{i*}(t)$ be the optimal control variables. If the following condition is satisfied, the Nash equilibrium exists [42].

$$H^i\left(x^*, a^{1*}, \dots, a^{(i-1)*}, a^{i*}, a^{(i+1)*}, \dots, a^{N*}, \lambda, t\right) \geq H^i\left(x^*, a^{1*}, \dots, a^{(i-1)*}, a^{i}, a^{(i+1)*}, \dots, a^{N*}, \lambda, t\right), \qquad t \in [0,T]$$

Definition 3 (Closed-loop game). The equilibrium is called a closed-loop Nash equilibrium if the equilibrium paths are a function of time and the current states of the system are presented as follows [41].

$$a^{i*}(x,t) = \phi^{i*}(x,t), \qquad i = 1, 2, \dots, N$$

The dependency of players' actions with the state variable *x* is examined whether the Nash strategy exists, hence,

$$\dot{\lambda}^i = -H_x^i - \sum_{j=1\,, j\neq i}^{N} H_{a^j}^i\, \phi_x^j$$

In the above HJB function, there are the following three steps to find the equilibrium paths of the game:

1- Let *r* be a fixed discount factor [4]. The HJB equation for each player is formulated as follows:

$$rV = \max[\, f(x(t), u(t); a) + \frac{\delta V}{\delta x}(g(x(t), u(t); a)) \tag{1}$$

where

$$V = \max\ [\int_0^\infty e^{-rt}(f(x(t), u(t); a)\, dt\,)]$$
$$s.t. \quad \dot{x}(t) = g(x(t), u(t); a) \tag{2}$$
$$x(t) = x_t$$

2- The first-order condition is used to check if the derivation for each player exists to build the Hessian matrix.

3- The second-order condition (first derivative test) is first expressed by building the Hessian matrix as follows.

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 V_i}{\partial x_2^2} \end{bmatrix} \tag{3}$$

And then the following conditions of equation (1) are checked in a presumptive differential game with two control variables and the HJB equation:

$$\begin{cases} (x_1^*, x_2^*) \text{ is } max\ \ f & if\ \ \frac{\partial^2 f}{\partial x_1^2} < 0 \ \ \&\ \ D(H) > 0 \\ (x_1^*, x_2^*)\ is\ min\ \ f & if\ \ \frac{\partial^2 f}{\partial x_1^2} > 0 \ \ \&\ \ D(H) > 0 \\ \text{no optimal solution} & \text{Otherwise} \end{cases} \qquad (4)$$

3.2. Nerlove and Arrow model

Let us briefly describe the N-A model here ([33]; [26]). The N-A model was presented by Nerlove and Arrow (1962) to analyse the impact of advertisement on demand over time and describe that the stream of advertising expenditure can be used to purchase the "goodwill" (also called the product image) that will maximize the firm's profit. Needless to say, "goodwill" can be human capital [6], health capital [22], or stock of durable goods leased to others [40]. The original N-A model is popular in dynamic advertising problems and has been widely used and extended by many researchers (see e.g. [2]; [10]). This model regards advertisement as an investment rather than a cost. Let $A(t)$ present the effect of past and current advertising on demand, which is known as goodwill. Assume that an increase in the current advertising expenditure leads to a rise in $A(t)$. Likewise, more expenditure on advertising in the past periods influences the current degree of $A(t)$ but with less contribution. Let us also assume that the current advertising expenditure rate, $u(t)$, is non-negative and δ is a deterioration factor of goodwill in the absence of advertisement. The respective state equation can be defined as follows:

$$\dot{A}(t) + \delta A(t) = u(t) \qquad (5)$$

where $\dot{A}(t) = \frac{dA}{dt}$ represents differentiation with respect to time. Equation (5) is a net investment in goodwill that can be computed by the difference between gross investment ($u(t)$) and the depreciated stock of goodwill. Nerlove and Arrow [33] argued that the quantity demanded at time t, $q(t)$, depends on the price charged $p(t)$, stock of goodwill $A(t)$, and uncontrollable variables $Z(t)$ such as customer income level, gender, and market features and regulations that might affect demand. Therefore, the demand function can be defined as $q(t) = f(p(t), A(t), Z(t))$. Moreover, the revenue of production expenses, $R(t)$, can be calculated using $R(t) = p(t)q(t) - C(q)$ where $C(q)$ denotes total production costs incurred to satisfy demand q. Let $r(t)$ be net revenue of production expenses and current advertising expenditure (also called profit); then $r(t) = R(t) - u(t)$. The N-A problem assumes that the firm's objective is to maximize the value of net revenue of production expenses and current advertising expenditure by seeking optimal price, p, and advertising strategies, u, over time.

Given the initial values of $A(0) = A_0$ and $P(0) = P_0$, the following model can be obtained optimal price and advertising policies over time:

$$V\{p, u, A\} = \max \int_0^{\infty} e^{-\alpha t} r(t)\, dt$$
$$s.t. \quad \dot{A}(t) = u(t) - \delta A(t)$$

where α is a constant interest rate.

3.3. Problem description and assumptions

In the differential game of a duopoly, two players (firms), $i \in \{1,2\}$, compete for the market share (on the same product [11]. In a closed-loop Nash equilibrium, it is assumed that both players are aware of the initial conditions and current states of the game over a continuous time horizon ($t \in (0, \infty)$) [7]. In the considered market, the desired product is consumed rapidly or is sold again by customers after consumption [19]. Assume that the constant production cost and initial state conditions are zero.

We assume that customers can decide to either continue the purchase from the current seller or shift their supplier [19]. However, customers cannot forecast the future prices of the firm products and their reaction is only for present prices [19]. Assume that the maximum likelihood of customers purchasing the products of firm *i* or *j* is denoted by v_i and v_j. Let s_i denote the cost of shifting of the i[th] firm which is a non-negative constant.

3.4. Mathematical model

We use the following notation and definitions:

Indices	Definitions
i, j	Two players (firms), *i*=1 and *j*=2
T	Time
Parameters	
β	Coefficient of price on demand
γ	Coefficient of goodwill on demand
δ	Constant depreciation rate of goodwill over time
$[L, M]$	Uniform distribution range for customers' purchase
r	Discount rate
s_i (s_j)	Shifting cost of firm *i (j)*
α_i (α_j)	Potential of the market for products of firm *i (j)*
m_i (m_j)	Marginal cost of production for firm *i (j)*
State variable	
A_i (A_j)	Goodwill for the product of firm *i (j)* by advertising

Control variable	
$p_i\ (p_j)$	Product price of firm *i (j)*
$u_i\ (u_j)$	Advertising rate of firm *i (j)*
Random variable	
$v_i\ (v_j)$	Probability of purchasing the products of firm *i (j)* by customers
Function	
$C(u_i)\ C(u_j)$	Advertising cost of the firm *i (j)*
$D_i\ (D_j)$	Demand for firm *i (j)*
$D_i'\ (D_j')$	Part of demand as a result of pricing and advertising decisions of firm *i (j)*
$\Delta D_i\ (\Delta D_j)$	Demand change due to shifting customers between firms *i* and *j*
$q_{ij}(q_{ji})$	Probability of a customer that is currently served by firm *i* (*j*) switches to firm *j* (*i*)
$\pi_i\ (\pi_j)$	Profit of firm *i (j)*
$V_i\ (V_j)$	Payoff of player *i (j)*

We assume that a customer of firm *j* shifts to firm *i* under the following condition ([15]; [19]).

$$v_i - p_i(t) - s_i > v_j - p_j(t) \tag{6}$$

where $p_i(t)\ and\ p_j(t)$ are the price of a product presented at time *t* by firm *i* and *j*, respectively.

In this game, the utility of player *i* is to maximise its profit which is reliant on pricing and advertising strategies. In other words, it warrants to maximise the following equation:

$$\pi_i(p_i(t), u_i(t), t) = (p_i(t) - m_i)D_i - C(u_i(t))$$

where $C(u_i(t)) = \frac{c_i}{2}u_i(t)^2$ is the advertising cost function of player *i*. Transferring the customers between the firms leads to the change of demand change as $D_i = D_i' + \Delta D_i$ where D_i' is a factor associated with the demand function of player *i* which is dependent on pricing and advertising strategy, and ΔD_i represents the change in demand of player *i* at each time when transferring the players between firms. Furthermore, the effect of current advertising and goodwill on future demand is defined as:

$$\dot{A}_i(t) = \frac{dA_i}{dt} = u_i(t) - \delta A_i(t) \tag{7}$$

We determine demand as a linear function of the price and goodwill at time *t* as given below.

$$D_i'(p_i(t), A_i(t), t) = \alpha_i + \beta p_i(t) + \gamma A_i(t);\ \beta < 0, \gamma > 0$$

Given equation (6), the probability of the customers' movement from firm *j* to *i*, q_{ji}, can be expressed as follows:

$$q_{ji} = Pr(v_i - v_j > p_i(t) - p_j(t) + s_i)$$

Similarly, the probability of the customers' movement from firm *i* to *j*, q_{ij}, can be defined as follows:

$$q_{ij} = Pr(v_j - v_i > p_{j(t)} - p_i(t) + s_j)$$

In addition, we assume that $v_i - v_j$ is uniformly distributed in $[L, M]$ where v_i and v_j are the probability of purchasing the products of firms *i* and *j* by customers [19]. Accordingly, the cumulative probability function (CPD) of q_{ji} can be defined as follows:

$$F_{q_{ji}}(x) = \begin{cases} 0 & if \quad x < L \\ \frac{x-L}{M-L} & if \quad L \leq x \leq M \\ 1 & if \quad x > M \end{cases}$$

where $x = p_i(t) - p_j(t) + s_i$, and the CPD of q_{ij} is:

$$F_{q_{ji}}(x) = \begin{cases} 0 & if \quad x < L \\ \frac{x-L}{M-L} & if \quad L \leq x \leq M \\ 1 & if \quad x > M \end{cases}$$

where $x = p_j(t) - p_i(t) + s_i$. Consequently, the change in demand associated with player *i* is computed as $\Delta D_i = D_j'(q_{ji} - q_{ij})$. Therefore, the demand function of player *i* is given as:

$$D_i = \alpha_i - \beta p_i(t) + \gamma A_i(t) + D_j'(q_{ji} - q_{ij})$$

In view of the above demand function, D_i, the value function of player *i* is expressed as follows:

$$V_i = \int_0^\infty e^{-rt}((p_i(t) - m_i)D_i - C(u_i(t)))\, dt$$

$$\begin{aligned} s.t: \quad & \dot{A}_i(t) = u_i(t) - \delta A_i(t) \\ & A_i(0) = 0\, ;\, u_i(0) = u_0\, ;\, p_i(0) = p_0 \\ & u_i(t) \geq 0 \end{aligned} \tag{8}$$

Note that the above discussions can be straightforwardly extended to player *j*.

Theorem. The equilibrium path for advertising rate and the price for firm *i* is determined by $u_i^*(t) = \frac{\frac{\partial Z_i}{\partial A_i}}{c_i}$ and $p_i^*(t) = \frac{F}{2H}$, respectively, where $F = (4m_i + 2m_j - s_i + s_j + 2L\alpha_i + La_j - 2M\alpha_i - M\alpha_j + L^2\beta^2 m_i + M^2\beta^2 m_i + 2\gamma LA_i(t) + \gamma LA_j(t) - 2\gamma MA_i(t) - \gamma MA_j(t) - 4\beta Lm_i - \beta Lm_j + 4\beta Mm_i + \beta Mm_j + \beta Ls_i - \beta Ls_j - \beta Ms_i + \beta Ms_j - \alpha_i\beta L^2 - \alpha_i\beta M^2 - \beta\gamma L^2 A_i(t) - \beta\gamma M^2 A_i(t) - 2ML\beta^2 m_i + 2ML\beta\alpha_i + 2ML\beta\gamma A_i(t))$ and $H = 2(\beta^2 L^2 - 2ML\beta^2 - 4L\beta + M^2\beta^2 + 4M\beta + 3)$.

Proof. To reach the price equilibrium path for player *i,* it is of the essence to examine the HJB conditions, i.e., equations (1) to (4). To do so, due to high complexity, a MATLAB code is provided and the derivative results of V_1 and V_2 concerning $U_1(t)$, $U_2(t)$, $P_1(t)$, and $P_2(t)$ are determined. Accordingly, the optimal trajectories for price and advertising for player *i,* the HJB equation of the value function for the respective player in problem (equation 8) is defined as follows:

$$rV_i = max[Z] = max\left[(p_i(t) - m_i)D_i - C(u_i(t) + \frac{\partial V_i}{A_i(t)}(u_i(t) - \delta A_i(t))\right]$$

where $D_i = \alpha_i - \beta p_i(t) + \gamma A_i(t) + q_{ji} D'_j - q_{ij} D'_i = \alpha_i - \beta p_i(t) + \gamma A_i(t) + (\frac{M - p_i(t) + p_j(t) - s_i}{M-L}) D'_j - (\frac{M - p_j(t) + p_i(t) - s_j}{M-L}) D'_i$. To maximize Z, the existence of the first-order condition is first checked by means of calculating the partial derivative of Z_i with respect to p_i and u_i as presented below.

$$\frac{\partial Z_i}{\partial p_i} = \alpha_i + \gamma A_i(t) - \beta p_i(t) + m_i p_i(t) \left(\beta + \frac{2}{L-M} + \frac{M - p_i(t) + p_j(t) - s_i}{L-M} - \frac{M + p_i(t) - p_j(t) - s_j}{L-M}\right)$$

$$\frac{\partial Z_i}{\partial u_i} = \frac{\partial V_i}{A_i(t)} - C(u_i(t))$$

Then, the Hessian matrix for a given player *i* in a duopoly market is expressed as $H(Z_i) = \begin{bmatrix} \frac{\partial^2 V Z_i}{\partial p_i^2(t)} & \frac{\partial^2 Z_i}{\partial p_i(t)\partial u_i(t)} - \frac{\partial^2 Z_i}{\partial u_i(t)\partial p_i(t)} & \frac{\partial^2 Z_i}{\partial u_i^2(t)} \end{bmatrix}$ to check the second-order condition. The following conditions need to be fulfilled to find the equilibrium path:

1. $\frac{\partial^2 Z_i}{\partial p_i^2(t)} = -2\beta - \frac{4}{L-M} < 0$
2. $D\big(H(Z_i)\big) = \left(\frac{\partial^2 Z_i}{\partial p_i^2(t)} \times \frac{\partial^2 Z_i}{\partial u_i^2(t)} - \frac{\partial^2 Z_i}{\partial p_i(t)\partial u_i(t)} \times \frac{\partial^2 Z_i}{\partial u_i(t)\partial p_i(t)}\right) = c_i \times \left(2\beta + \frac{4}{L-M}\right) > 0$

where $\frac{\partial^2 Z_i}{\partial u_i^2(t)} = -c_i$ and $\frac{\partial^2 Z_i}{\partial p_i(t)\partial u_i(t)} = \frac{\partial^2 Z_i}{\partial u_i(t)\partial p_i(t)} = 0$. Therefore, if $\beta > \frac{2}{M-L}$ the conditions will be satisfied. By solving the system of equations for $\{\frac{\partial Z_i}{\partial p_i} = 0, \frac{\partial Z_i}{\partial u_i} = 0, \frac{\partial Z_j}{\partial p_j} = 0, \frac{\partial Z_j}{\partial u_j} = 0\}$ the optimal trajectories for price and advertising are defined as $p_i^*(t)$ and $u_i^*(t)$. Notice that the focus of the equilibrium paths is limited to one player to avoid unnecessary repetition. In the light of $p_i^*(t)$ and $u_i^*(t)$, the maximum value of player *i* is presented as follows:

$$V_i^* = \frac{1}{r}[(p_i^*(t) - m_i)D_i^* - C(u_i^*(t)) + \frac{\partial V_i}{A_i(t)}(u_i^*(t) - \delta A_i(t))$$

where $D_i^* = \alpha_i + \beta p_i^*(t) + \gamma A_i(t) + q_{ji}\big(\alpha_j + \beta p_j^*(t) + \gamma A_j(t)\big) - q_{ij}\big(\alpha_i + \beta p_i^*(t) + \gamma A_i(t)\big)$.

4. Empirical Analysis and findings

This section aims to apply the proposed model to the cellular data market in the emerging economy of Iran using four numerical scenarios shown in Tables 3 to 6. Two main cellular carriers in Iran are MTN-IR as player1 and MCI as player 2. Having Similar network coverage and analogous and copiable bundles, the only variables they find attractive tend to be pricing and advertising decisions to increase their market demand and substantially their profit. That is why this market is a potential fit for the given model. Since the real data in the Iran Telecom market is often considered to be confidential, four numerical scenarios are demonstrated to investigate the empirical application of the respective model in the real world. The scenarios are designed using the consultant of experts from both carries MTN-IR and MCI as well as the related literature. However, the authors acknowledge that measuring an accurate amount of parameters needs more careful investigation. The changes in (s_i, s_j),(m_i, m_j), (c_i, c_j) and (α_i, α_j) parameters of players 1 and 2 and their effect on goodwill, optimal price and advertising trajectories, and the changes of companies profit are studied over an infinite time horizon. . For more clarification, we present the calculations of the first numerical example.

4.1. Analysing the effects of (s_i, s_j)

Table 3 presents the values of parameters for the two players (*i,j*) denoted by players 1 and 2.

Table 3. An example to analyse the effect of (s_i, s_j)

Parameters	Numerical values	Parameters	Numerical values
s_i	3	s_j	1
m_i	5	m_j	5
c_i	1	c_j	1
α_i	10	α_j	10
r	0.1	β	-2.5
γ	0.1	δ	0.1
L	0	M	1

The equilibrium paths of players 1 and 2 in relation to the price and advertising are calculated as follows.

$$p_1^* = \frac{9A_1(t)}{770} + \frac{A_2(t)}{385} + \frac{687}{154} \ and\ u_1^* = \frac{\partial V_1}{\partial A_1(t)}$$

$$p_2^* = \frac{A_1(t)}{385} + \frac{9A_2(t)}{770} + \frac{743}{174} \ and\ u_2^* = \frac{\partial V_2}{\partial A_2(t)}$$

Given the HJB, the Value functions of the players are presented below:

$$V_1 = 10\left(\frac{9A_1(t)}{770} - \frac{A_2(t)}{385} + \frac{83}{154}\right)\left(\frac{137A_1(t)}{1540} - \frac{19A_2(t)}{770} + \frac{37}{308}\right) - 10\frac{\partial V_1}{\partial A_1(t)}\left(\frac{A_1(t)}{10} - \frac{\partial V_1}{\partial A_1(t)}\right) - 5\left(\frac{\partial V_1}{\partial A_1(t)}\right)^2$$

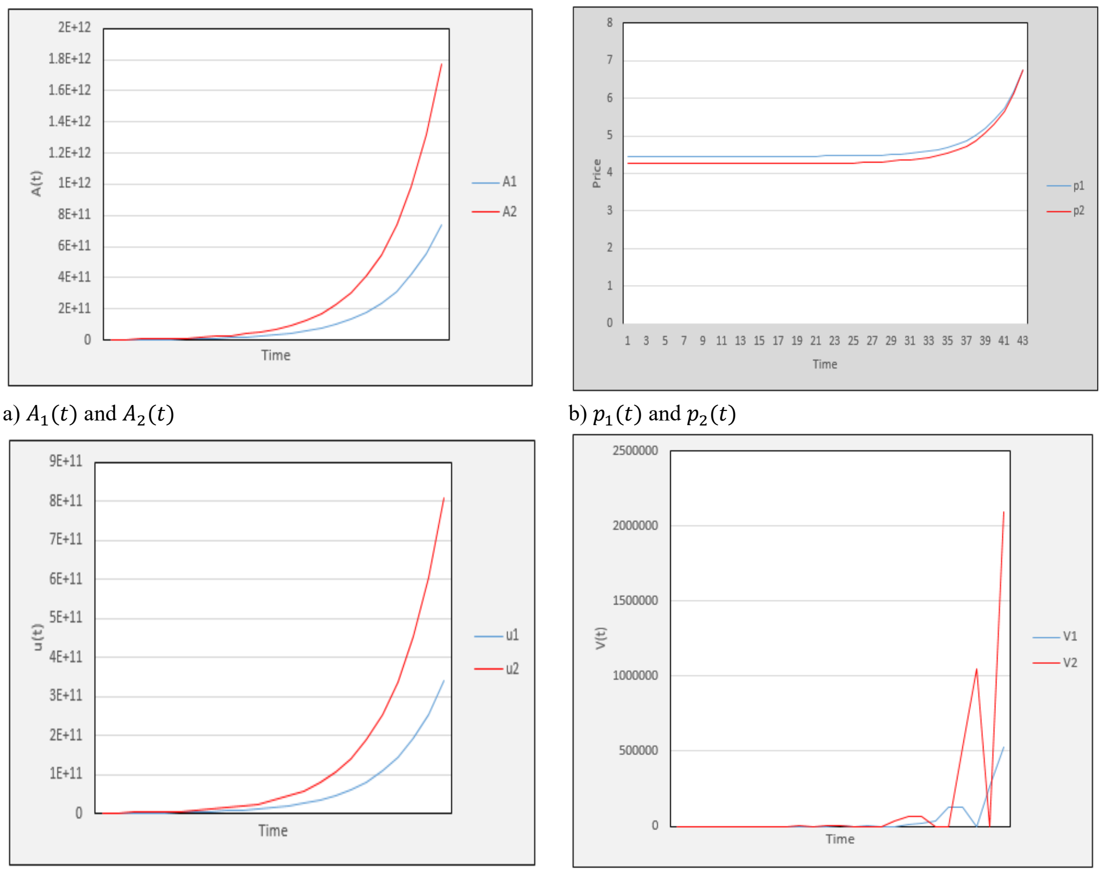
$$V_2 = -10\left(\frac{A_1(t)}{385} + \frac{9A_2(t)}{770} - \frac{27}{154}\right)\left(\frac{A_1(t)}{154} - \frac{109A_2(t)}{1540} + \frac{635}{308}\right) - 10\frac{\partial V_2}{\partial A_2(t)}\left(\frac{A_2(t)}{10} - \frac{\partial V_2}{\partial A_2(t)}\right) - 5\left(\frac{\partial V_2}{\partial A_2(t)}\right)^2$$

The first order derivation of the above equations for players *i* and *j* is presented as follows:

$$\frac{\partial V_1}{\partial A_1(t)} = \frac{A_1(t) + \left(\frac{4696A_1(t)^2}{5929} - \frac{68A_1(t)A_2(t)}{5929} - \frac{55190A_1(t)}{5929} + \frac{76A_2(t)^2}{5929} - \frac{16140A_2(t)}{5929} - \frac{76775}{5929}\right)^{\frac{1}{2}}}{10}$$

$$\frac{\partial V_2}{\partial A_2(t)} = \frac{A_2(t) + \left(\frac{20A_1(t)^2}{5929} - \frac{128A_1(t)A_2(t)}{5929} + \frac{5000A_1(t)}{5929} + \frac{4948A_2(t)^2}{5929} - \frac{43290A_2(t)}{5929} - \frac{428625}{5929}\right)^{\frac{1}{2}}}{10}$$

Considering equation (7), the above results are equal to $\frac{dA_1}{dt} + \delta A_1(t)$ and $\frac{dA_2}{dt} + \delta A_2(t)$ respectively. Accordingly, the optimal values for price, advertising cost, goodwill, and payoff are obtained by solving the above differential equations. Following the models and equations illustrated above, changes in $A_i(t)$, $p_i(t)$, $u_i(t)$ and V_i of both players are exhibited in Figures 1 with respect to different (s_i, s_j) of players.



a) $A_1(t)$ and $A_2(t)$

b) $p_1(t)$ and $p_2(t)$

c) $u_1(t)$ and $u_2(t)$

d) $V_1(t)$ and $V_2(t)$

Figure 1. The change of the variable with respect to (s_i, s_j) overtime for two players

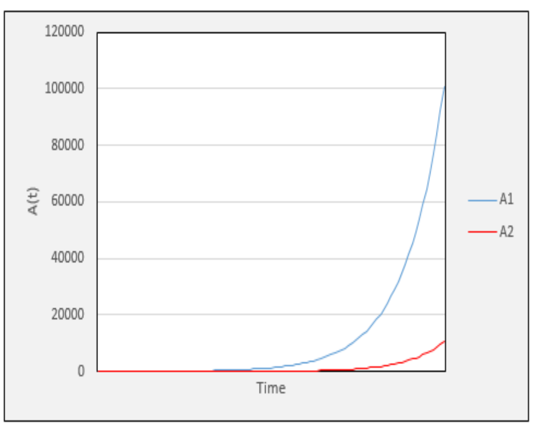
Figure 1(a) indicates that the goodwill of both players is increased over time; however, the MTN-IR with a higher shift cost attain lower amounts of goodwill, generally speaking, it is likely that customers to be dubious about player 1 as it makes an attempt to keep customers in captivity by a higher SC, which adversely affects its goodwill. It is a notion that companies should look for brand equity as their ultimate goal, they need to decrease their SC. Figure 1(b) shows that the MTN-IR with a higher shift cost leads to the higher price by virtue of customer lock-in; nonetheless, after a while, the MTN-IR has to reduce the price in comparison with MCI. That is to say, while in short term higher SC leads to a higher level of price, it cannot be a long-term solution if companies are inclined toward gaining profit through high prices. Figure 1(c), represents the direct relationship between the firm's advertising and goodwill over time. Figure 1(d) demonstrates that a higher price and advertising do not necessarily result in more profit for the company. Moreover, the cumulative payoff function of the second firm is more than the competitor, and the firm with a higher shift cost will gain profit less than the rival. This finding is opposed to [20] that firms with higher SC can enjoy profitable market power.

4.2. Analysing the effects of (m_i, m_j)

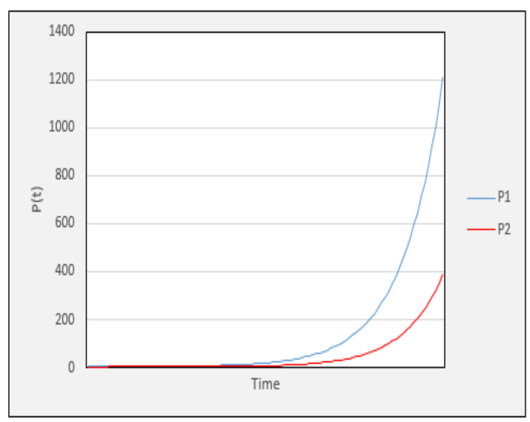
Given Table 4, the results for $A_i(t)$, $p_i(t)$, $u_i(t)$ and V_i with respect to different (m_i, m_j) are portrayed in Figure 2.

Table 4. An example to analyse the effect of (m_i, m_j)

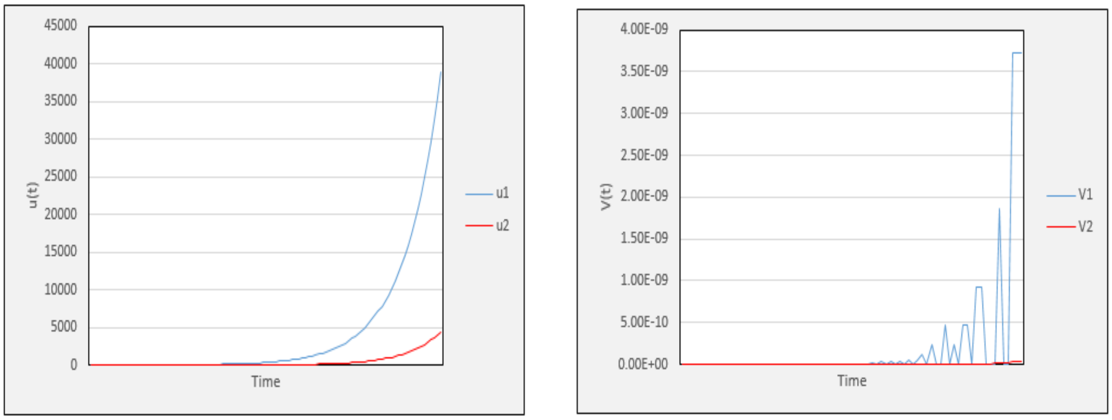
Parameters	Numerical values	Parameters	Numerical values
s_i	1	s_j	1
m_i	5	m_j	2
c_i	1	c_j	1
α_i	10	α_j	10
r	0.1	β	-2.5
γ	0.1	δ	0.1
L	0	M	1



a) $A_1(t)$ and $A_2(t)$



b) $p_1(t)$ and $p_2(t)$



c) $u_1(t)$ and $u_2(t)$ d) $V_1(t)$ and $V_2(t)$

Figure 2. The change of the variable with respect to (m_i, m_j) overtime for two players

Figure 2(a) shows the direct relationship between goodwill and marginal cost. It is obvious that the goodwill of both players increases but the exponential growth of the MTN-IR is viewed over time. Figure 2(b) displays that the player with a higher marginal cost has a higher equilibrium price. Furthermore, the player with a higher marginal cost has a higher advertising rate as depicted in Figure 2(c). The advertising of both players rises but it is extremely higher for the MTN-IR. Figure 2(d) indicates that players' profit is oscillating up and down over time and the oscillation of the first player is more significant. This occurrence can be interpreted in the way that the first player boosts up to the level of price and advertisement cost to cover the higher production cost. This leads to more competition over time and this situation causes a negative effect for both competitors. Notably, although the overall profit of the MTN-IR is more than the MCI, this amount is less than the first scenario indicating the negative effect of higher production costs. On the other hand, the profit of MCI fluctuates in the vicinity of zero which means it is highly likely that player 2 never begins this game; therefore, the market will turn into a monopoly.

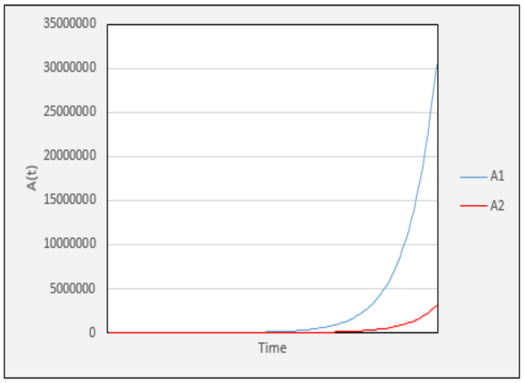
4.3. Analysing the effects of (c_i, c_j)

Let us focus on the dataset in Table 5 to analyse the effects of (c_i, c_j) for two players.

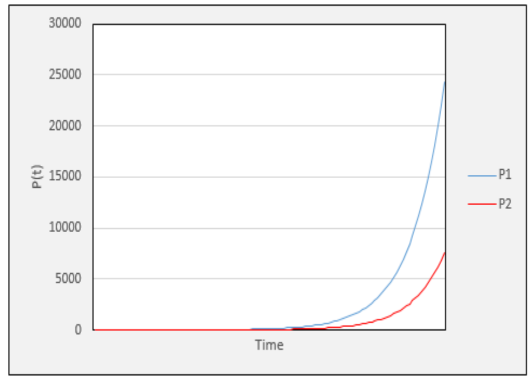
Table 5. An example to analyse the effect of (C_i, C_j)

Parameters	Numerical values	Parameters	Numerical values
s_i	1	s_j	1
m_i	5	m_j	5
c_i	3	c_j	1
α_i	10	α_j	10
r	0.1	β	-2.5
γ	0.1	δ	0.1
L	0	M	1

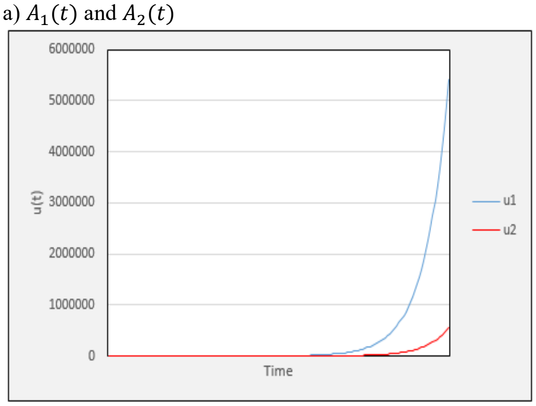
Figures 3 represents the results for $A_i(t)$, $p_i(t)$, $u_i(t)$ and V_i after one applies the proposed models in this paper with respect to different (C_i, C_j) of players.



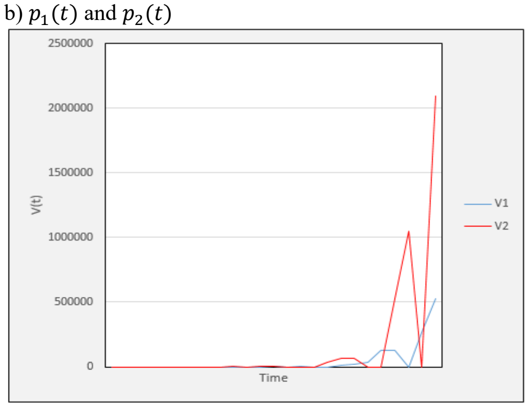
a) $A_1(t)$ and $A_2(t)$



b) $p_1(t)$ and $p_2(t)$



c) $u_1(t)$ and $u_2(t)$



d) $V_1(t)$ and $V_2(t)$

Figure 3. The change of the variable with respect to (C_i, C_j) overtime for two players

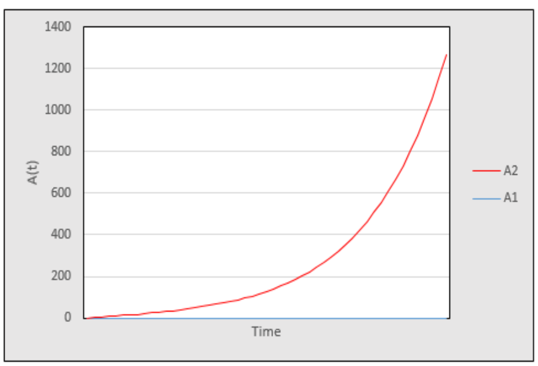
Since $c_1 > c_2$ and $A_1(t) > A_2(t)$, we can conclude that MTN-IR that employs higher advertisement cost gains higher goodwill over time as shown in Figure 3(a) (which is obvious). Furthermore, players' goodwill is on the rise but the goodwill growth of MTN-IR is more rapid than that of MCI. The initial equilibrium prices are almost equal while MTN-IR 's price increases quicker than that of MCI over time (see Figure 3(b)). Notably, based upon the numerical example and scenario analysis, the difference in the prices is not more than 3 times which means higher advertising costs will not necessarily lead to a dramatic price gap in the market. Figure 3(c) reveals that even at the moment of observing the low-price difference, MTN-IR needs to pay more for advertising due to using more expensive advertising tools. Figure 3(d) signifies that the profit rate of both players is very low at first. Then the oscillations of the profit for both players occur wherein they are more considerable in MTN-IR. Importantly, MTN-IR should not leave the game even if he gets lost in some ranges because ultimately the overall profit of the first firm is higher than the MCI. This is an important case for companies with high advertising investment, which proves such investments will lead to a higher profit compared to the rival in the long term, while in the short-term the company cannot be lucrative.

4.4. Analysing the effects of (α_i, α_j)

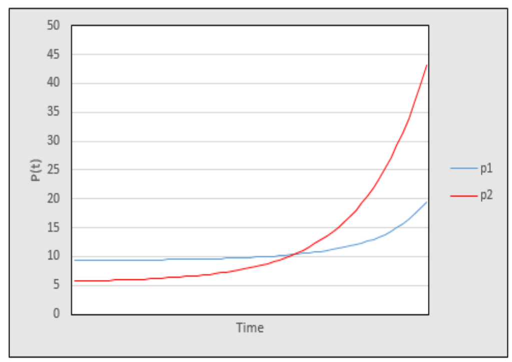
The last numerical example is based on the dataset in Table 6 with the aim of analysing the effect of (α_i, α_j) for two players. The led changes in $A_i(t)$, $p_i(t)$, $u_i(t)$ and V_i are portrayed in Figure 4.

Table 6. An example to analyse the effect of (α_i, α_j)

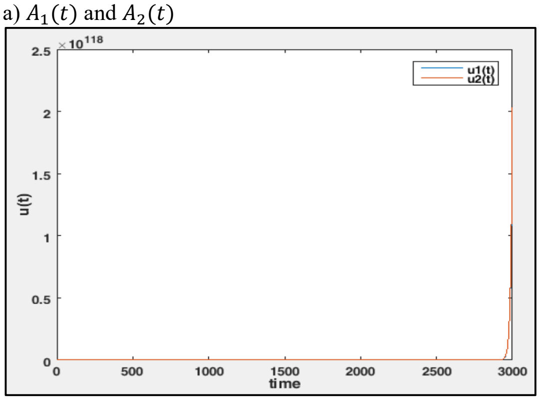
Parameters	Numerical values	Parameters	Numerical values
s_i	1	s_j	1
m_i	5	m_j	5
c_i	1	c_j	1
α_i	50	α_j	10
r	0.1	β	-0.5
γ	0.1	δ	0.1
L	1	M	2



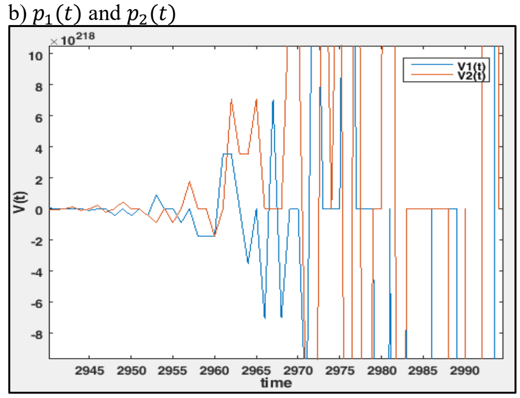
a) $A_1(t)$ and $A_2(t)$



b) $p_1(t)$ and $p_2(t)$



c) $u_1(t)$ and $u_2(t)$



d) $V_1(t)$ and $V_2(t)$

Figure 4. The change of the variable with respect to (α_i, α_j) overtime for two players

MTN-IR with a higher initial potential to enter the market (α_i) does not lead to more goodwill (see Figure 4(a)). Moreover, the goodwill level and growth over time are higher for player 2 that has less potential to enter the market. Figure 4(b) demonstrates that MTN-IR with a higher intention to enter the market has a greater initial equilibrium price. Appealingly, MTN-IR is dominated by the MCI over time. Referring to Figure 4(c), the equilibrium level of advertising is the same for both players and is not affected by the potential to enter the market. Figure 4(d) indicates that the firms' profit is trivial at the beginning but the oscillations are observed over time. Contrary to previous examples, the oscillations of both firms are almost identical. Moreover, the overall profit for both firms is nearly equal, emphasizing that the potential to enter a duopoly market does not affect the profit of a firm, and the high market potential of player 1 can be paid back by the higher goodwill of player 2.

5. Conclusions

The dynamic duopoly pricing and advertising model developed here allows us to expand our ability in studying joint pricing and advertising decisions in the presence of shift cost. In the duopoly market, there are two players only, and the decisions of a player influence strongly the decisions and profit of the other player. Therefore, proper pricing and advertising strategies are of paramount importance to maximise their gains over time. The chosen optimal price can oscillate in a range that leads to the fulfillment of demands, profit maximization, and customer satisfaction. However, establishing optimal pricing and advertising strategies is a challenge for the financial and marketing teams of a firm. The significant contribution of this research is considering the impact of shifting costs on firms' demand. To this end, the dynamic pricing and advertising model of Nerlove-Arrow (N-A) is adopted in the presence of the shift cost (SC) between two players which can be contributed to two fields of pricing and marketing. In this regard, a differential game theory approach is suggested to seek equilibrium trajectories of price and advertising over time.

Beyond its methodological contributions, the model can be applied empirically. For this reason, four numerical scenarios are presented for two main players of cellular carriers in the Iran Telecom industry with similar premises to the model and analysed the result of the proposed model in terms of the effects of shifting cost, the initial potential to enter the market, production cost and margin costs of the firms on goodwill, price, advertising, and profit. The findings of these scenarios are summarized in Table 7. Furthermore, the most lucrative scenarios for player 1 are detected as scenario 1 > scenario 3 > scenario 4 > scenario 2, respectively. That is to say, in the case of the presence of SC, although the cumulative earned profit of MTN-IR is lower than its rival, the total profitability of the market is shifted up; therefore, using SC by one of the players will affect the entire market positively. On the other hand, surprisingly, the high marginal cost of production for one of the players will destroy the profitability of the entire market through the high pricing and advertising competition.

Table 7. Summary of scenario analysis results

	$A_i(t)$	$p_i(t)$	$u_i(t)$	V_i
$s_1 > s_2$	$A_1(t)\uparrow,\ A_2(t)\uparrow$ $A_1(t) > A_2(t)$	$p_1(0) > p_2(0) > 0$ $p_1(t)\uparrow,\ p_2(t)\uparrow$ $p_1(t) = p_2(t)\ : for\ t \to \infty$	$u_1(0) = u_2(0) = 0$ $u_1(t)\uparrow, u(t)\uparrow$ $u_1(t) > u_2(t)$	$V_1(0) = V_2(0) = 0$ $\sum_{i=0}^{\infty} V_1(t) < \sum_{i=0}^{\infty} V_2(t)$
$m_1 > m_2$	$A_1(t)\uparrow,\ A_2(t)\uparrow$ $A_1(t) > A_2(t)$	$p_1(0) = p_2(0) = 0$ $p_1(t)\uparrow,\ p_2(t)\uparrow$ $p_1(t) \gg p_2(t)$	$u_1(0) = u_2(0) = 0$ $u_1(t)\uparrow, u(t)\uparrow$ $u_1(t) \gg u_2(t)$	$\sum_{i=0}^{\infty} V_1(t) = 0^+$ $\sum_{i=0}^{\infty} V_2(t) \approx 0$
$C_1 > C_2$	$A_1(t)\uparrow, 5\ A_2(t)\uparrow$ $A_1(t) > A_2(t)$	$p_1(0) = p_2(0) = 0$ $p_1(t)\uparrow,\ p_2(t)\uparrow$ $p_1(t) > p_2(t)$ $\frac{p_1(t)}{p_2(t)} < 3$	$u_1(0) = u_2(0) = 0$ $u_1(t)\uparrow, u(t)\uparrow$ $u_1(t) > u_2(t)$	$V_1(0) = V_2(0) = 0$ $\sum_{i=0}^{\infty} V_1(t) > \sum_{i=0}^{\infty} V_2(t)$
$\alpha_1 > \alpha_2$	$A_1(t) = 0$ $A_2(t)\uparrow$	$p_1(0) > p_2(0)$ $p_1(t)\uparrow,\ p_2(t)\uparrow$ $\begin{cases} p_1(t) > p_2(t)\ if \quad t < T \\ p_1(t) < p_2(t)\ if \quad t > T \end{cases}$	-	$V_1(0) = V_2(0) = 0$ $\sum_{i=0}^{\infty} V_1(t) \approx \sum_{i=0}^{\infty} V_1(t)$

The proposed model can potentially lend itself to other types of markets such as oligopoly and full competition. We also hope that the Nerlove and Arrow model introduced in this paper provides the groundwork for comparing this with other methods frequently used in the literature such as Vidale and Wolfe model [36], Lanchester models [29], dynamic advertisements and competition models [24] and diffusion models [44].

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