Average-case consistency measurement and analysis of interval-valued reciprocal preference relations

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Abstract

Measuring consistency of preferences is very important in decision-making. This paper addresses this key issue for interval-valued reciprocal preference relations. Existing studies implement one of two different measures: the "classical" consistency measure, and the "boundary" consistency measure. The classical consistency degree of an interval-valued reciprocal preference relation is determined by its associated reciprocal preference relation with highest consistency degree, while the boundary consistency degree is determined by its two associated boundary reciprocal preference relations. However, the consistency index of an interval-valued reciprocal preference relation should be determined by taking into account all its associated reciprocal preference relations. Motivated by this, a new consistency measure for interval-valued reciprocal preference relations, the average-case consistency measure, is suggested and introduced. The new average-case consistency measure of an interval-valued reciprocal preference relation is determined as the average consistency degree of all reciprocal preference relations associated to the interval-valued reciprocal preference relation. Furthermore, the analysis and comparison of the different consistency measure internal mechanisms is used to justify the validity of the average-case consistency measure. Finally, an average-case consistency improving method which aims to obtain a modified interval-valued reciprocal preference relation with a required average consistency degree is developed.

Keywords: Decision analysis, reciprocal preference relation, interval-valued preferences, consistency measurement, average-case consistency.

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1. Introduction

Reciprocal preference relations are based on the pairwise comparison method, and are widely used preference representation structures in decision-making problems. Various types of reciprocal preference relations have been proposed, such as additive preference relations (also called fuzzy preference relations) [2, 11, 13, 14, 24], and multiplicative preference relations [3, 21–23]. It is well known that

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quantifying consistency is a very important issue in decision-making with preference relations. The lack of consistency can lead to inconsistent conclusions. In the specialised literature, a number of consistency measurement methods of reciprocal preference relations have been proposed (see, among others, [1, 7, 15, 17, 18, 34, 38]).

However, due to the complexity and uncertainty involved in real-world decision problems, it is sometimes unrealistic to acquire exact judgments. Thus, reciprocal preference relations are extended to interval-valued reciprocal preference relations (see, among others, [27, 35]). In this paper, we focus on the consistency of interval-valued reciprocal preference relations. Existing studies regarding the measurement of consistency of interval-valued reciprocal preference relations can be broadly classified as implementing one of two different measures that we refer to as: the "classical" consistency measure [8, 12, 27, 35], and the "boundary" consistency measure [19, 20], which are described in Section 2.2. However, based on the definitions of the classical and boundary consistency measures (see Eqs. (3) and (4)), we can find that:

(1) The classical consistency degree of an interval-valued reciprocal preference relation is determined by its associated reciprocal preference relation with highest consistency degree, while

(2) The boundary consistency degree is determined by its two associated boundary reciprocal preference relations.

It is natural that the consistency index of an interval-valued reciprocal preference relation should be determined by taking into account all its associated reciprocal preference relations. Motivated by this, in this paper a new average-case consistency analysis of interval-valued reciprocal preference relations is suggested, defined and analysed. Furthermore, this paper also proposes an average-case consistency improving method, based on the relationship among the average-case consistency measure, the classical consistency measure, and the worst consistency measure.

The rest of the paper is organised as follows. Section 2 introduces a basic description of the interval-valued reciprocal preference relation, the classical consistency measure and the boundary consistency measure. Section 3 presents the average-case consistency analysis of the interval-valued reciprocal preference relation (Section 3.1), as well as a numerical analysis (Section 3.2) and the different consistency measure internal mechanisms (Section 3.3). Section 4 is dedicated to the average-case consistency improving method. Finally, concluding remarks are included in Section 5.

2. Preliminaries

This section provides the basic knowledge regarding interval-valued reciprocal preference relations, as well as the classical consistency measure and the boundary consistency measure for interval-valued reciprocal preference relations.

2.1. Interval-Valued Reciprocal Preference Relations

The definitions of both the additive reciprocal preference relation and the interval-valued additive reciprocal preference relation are given below.

***Definition* 1 (Additive Reciprocal Preference Relation [13, 23]).** A matrix $F = (f_{ij})_{n\times n}$, with $f_{ij} \in [0,1]$ and $f_{ij} + f_{ji} = 1\,\forall i,j \in \{1,2,\ldots,n\}$, is called an additive reciprocal preference relation.

***Definition* 2 (Interval-Valued Additive Reciprocal Preference Relation [35]).** A matrix $\tilde{V} = (\tilde{v}_{ij})_{n\times n}$, with $\tilde{v}_{ij} = [v_{ij}^-, v_{ij}^+] \subseteq [0,1]$ and $v_{ij}^- + v_{ji}^+ = 1\,\forall i,j \in \{1,2,\ldots,n\}$, is called an interval-valued additive reciprocal preference relation.

There are two main types of reciprocal preference relations: additive reciprocal preference relations and multiplicative reciprocal preference relations. The transformation functions between these types of reciprocal preference relations have been presented in [3], so this paper focuses entirely on interval-valued additive reciprocal preference relations, and the proposed results can be similarly applied to interval-valued multiplicative reciprocal preference relations via the corresponding transformation function. In this paper, the additive reciprocal preference relation and the interval-valued additive reciprocal preference relation will be denoted simply as RPR and IVRPR, respectively.

Clearly the concept of IVRPR extends the concept of RPR, and when $v_{ij}^- = v_{ij}^+\ \forall i,j$ then an IVRPR becomes an RPR. However, when there exists at least a pair of values (i,j) such that $v_{ij}^- < v_{ij}^+$ then an IVPR can be seen as a collection of (associated) RPRs as the following definition implies:

***Definition* 3 (RPRs associated to an IVRPR [8]).** Let $\tilde{V} = (\tilde{v}_{ij})_{n\times n}$, with $\tilde{v}_{ij} = [v_{ij}^-, v_{ij}^+] \subseteq [0,1]$, be an IVRPR. An RPR $F = (f_{ij})_{n\times n}$ that verifies

$$v_{ij}^- \le f_{ij} \le v_{ij}^+ \quad \forall i,j \in \{1,2,\ldots,n\}$$

is called an RPR associated to $\tilde{V}$. The set of all RPRs associated to an IVRPR $\tilde{V}$ is denoted by $N_{\tilde{V}}$.

Notice that given an IVRPR $\tilde{V} = (\tilde{v}_{ij})_{n\times n}$, with $\tilde{v}_{ij} = [v_{ij}^-, v_{ij}^+] \subseteq [0,1]$, $N_{\tilde{V}}$ contains the following two associated RPRs $B = (b_{ij})_{n\times n}$ and $C = (c_{ij})_{n\times n}$:

$$b_{ij} = \begin{cases} v_{ij}^+ & i<j \\ 0.5 & i=j \\ v_{ij}^- & i>j \end{cases}, \quad c_{ij} = \begin{cases} v_{ij}^- & i<j \\ 0.5 & i=j \\ v_{ij}^+ & i>j \end{cases}. \tag{1}$$

In this paper the RPRs B and C are called the boundary RPRs associated to the IVRPR $\tilde{V}$.

2.2. Consistency Measures of IVRPRs

In the following, we provide the definition of the consistency index of an RPR. The classical and boundary consistency measures of an IVRPR that, based on the consistency index of RPRs, have been proposed in the literature are also provided.

Consistency index of RPRs [15]. Based on the additive transitivity property [24], Herrera-Viedma et al. [15] proposed the following consistency index (CI) of an RPR F:

$$CI(F) = 1 - \frac{4}{n \cdot (n-1) \cdot (n-2)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} |f_{ij} + f_{jk} - f_{ik} - 0.5| \tag{2}$$

The larger the value of $CI(F)$ the more consistent F is. Generally, in practice decision makers may establish a consistency threshold $\overline{CI}$ for RPRs so that a given RPR F that verifies $CI(F) \geq \overline{CI}$ is considered of acceptable consistency; otherwise, F is considered of unacceptable consistency.

Meanwhile, to our knowledge, two kinds of consistency measures for IVRPRs have been presented based on the concept of consistency index of an RPR given in Eq.(2):

Classical consistency measure of IVRPRs [12, 35]. Let $\tilde{V}$ be an IVRPR. If there exists an RPR associated to $\tilde{V}$, $F \in N_{\tilde{V}}$, such that $CI(F) = 1$ then $\tilde{V}$ is considered to be consistent. In this paper, the classical consistency index (CCI) of an IVRPR $\tilde{V}$ is formally expressed as follows:

$$CCI(\tilde{V}) = \max_{F \in N_{\tilde{V}}} CI(F) \tag{3}$$

Therefore, when $CCI(\tilde{V}) = 1$, $\tilde{V}$ is consistent; otherwise, $\tilde{V}$ is not consistent.

Boundary consistency measure of IVRPRs [19, 20]. Let $\tilde{V}$ be an IVRPR. If its associated boundary RPRs given in Eq.(1), B and C, are both of acceptable consistency, then $\tilde{V}$ is of acceptable consistency. In other words, if $CI(B) \geq \overline{CI}$ and $CI(C) \geq \overline{CI}$, then $\tilde{V}$ is of acceptable consistency; otherwise, $\tilde{V}$ is of unacceptably consistency. In this paper, the boundary consistency index (BCI) of an IVRPR $\tilde{V}$ is formally expressed as follows:

$$BCI(\tilde{V}) = \Big[\min\big\{CI(B), CI(C)\big\}, \max\big\{CI(B), CI(C)\big\}\Big]. \tag{4}$$

Based on Eqs.(3) and (4), it is easy to see that both the CCI and the BCI do not implement the consistency degree of all the RPRs associated to an IVRPR, and as such might not reflect the consistency of an IVRPR accurately. This argument is used and exploited in the following sections to propose a new type of consistency measure for IVRPRs, which is called the average-case consistency measure of IVRPRs, and that is determined as the average consistency degree of all associated RPRs to the IVRPR.

3. Average-case Consistency Measure of IVRPRs

This section proposes the average consistency index (ACI) of IVRPRs, followed by numerical examples and a comparative study to justify the feasibility of the new ACI to measure consistency of IVRPRs.

3.1. Average Consistency Index of IVRPRs

Let $\tilde{V} = ([v_{ij}^-, v_{ij}^+])_{n\times n}$ be an IVRPR. The underlying idea of the new proposed average-case consistency measure consists in measuring the consistency degree of an IVRPR using the average consistency of all its associated RPRs. Indeed, associated RPRs of $\tilde{V}$ can be represented by $F = (f_{ij})_{n\times n} \in N_{\tilde{V}}$, and $f_{ij}(i < j)$ can be considered as a random variable taking values in $[v_{ij}^-, v_{ij}^+]$. These assumptions are used to propose the following formal definition of the ACI of the IVRPR $\tilde{V}$:

***Definition* 4 (IVRPR Average Consistency Index).** Let $\tilde{V} = ([v_{ij}^-, v_{ij}^+])_{n\times n}$ be an IVRPR. The average consistency index (ACI) of $\tilde{V}$ is

$$ACI(\tilde{V}) = E(CI(F)), \tag{5}$$

where $F = (f_{ij})_{n\times n} \in N_{\tilde{V}}$ is the random RPR associated to $\tilde{V}$, i.e. f_{ij} are random variables in $[v_{ij}^-, v_{ij}^+]$ such that $f_{ji} = 1 - f_{ij}\ \forall i < j \in \{1, 2, \ldots, n\}$; $CI(F)$ is the consistency index random variable obtained via expression (2); and $E(CI(F))$ is the expected value of $CI(F)$. Consequently, we have

$$ACI(\tilde{V}) = 1 - \frac{4}{n\cdot(n-1)\cdot(n-2)} \sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=j+1}^{n} E\big(|f_{ij} + f_{jk} - f_{ik} - 0.5|\big). \tag{6}$$

The value $ACI(\tilde{V})$ measures on average the consistency degree of all the RPRs associated to $\tilde{V}$. Thus, the larger the value of $ACI(\tilde{V})$, the more consistent $\tilde{V}$ is.

The normal distribution is one of the most widely used probability distributions [30]. When a random variable X is distributed normally with mean u and variance σ^2, it is denoted by $X \sim N(u, \sigma^2)$ and its density function is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\dfrac{(x-u)^2}{2\sigma^2}}; \quad -\infty < x < \infty \tag{7}$$

In this paper, we assume that

$$f_{ij} \sim N(u_{ij}, \sigma_{ij}^2), \tag{8}$$

where

$$u_{ij} = \frac{v_{ij}^- + v_{ij}^+}{2}, \tag{9}$$

and

$$\sigma_{ij} = \frac{v_{ij}^+ - v_{ij}^-}{6}. \tag{10}$$

These assumptions (Eqs. (8)-(10)) are based on the following reasons:

(1) Based on Jong [16] and Dong et al. [10], decision makers often have certain consistency tendency in making pairwise comparisons, so in what follows it is assumed that $f_{ij}(i < j)$ relatively centralizes the domain close to $\frac{v_{ij}^- + v_{ij}^+}{2}$ and has a normal distribution, i.e., it is assumed that $f_{ij} \sim N(u_{ij}, \sigma_{ij}^2)$, where $u_{ij} = \frac{v_{ij}^- + v_{ij}^+}{2}$;

(2) The probability of f_{ij} distributed in the interval $[v_{ij}^-, v_{ij}^+]$ should be close to 1. According to the 3σ principle of normally distributed variables [30], it is known that $P(u_{ij}-3\sigma_{ij} \leq f_{ij} \leq u_{ij}+3\sigma_{ij}) \approx 1$. Because $u_{ij} = \frac{v_{ij}^- + v_{ij}^+}{2}$, $u_{ij} - 3\sigma_{ij} = v_{ij}^-$ and $u_{ij} + 3\sigma_{ij} = v_{ij}^+$, consequently, it is $\sigma_{ij} = \frac{v_{ij}^+ - v_{ij}^-}{6}$.

The following result derives from the well known statistical result regarding independent and identically distributed random variables applied to the normal distribution type [30]:

***Lemma* 1.** *Let $X \sim N(u_X, \sigma_X^2)$, $Y \sim N(u_Y, \sigma_Y^2)$ and $R \sim N(u_R, \sigma_R^2)$ be independent, and $Z = X + Y - R - 0.5$. Then it is $Z \sim N(u_Z, \sigma_Z^2)$, where $u_Z = u_X + u_Y - u_R - 0.5$ and $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 + \sigma_R^2$.*

According to Lemma 1 and the assumption, $f_{ij} \sim N\left(\frac{v_{ij}^- + v_{ij}^+}{2}, \left(\frac{v_{ij}^+ - v_{ij}^-}{6}\right)^2\right)$, we have the following main result, which provides the analytical procedure to compute the ACI of an IVRPR $\tilde{V} = (\tilde{v}_{ij})_{n\times n}$:

***Theorem* 1.** *Let $\tilde{V} = ([v_{ij}^-, v_{ij}^+])_{n\times n}$ be an IVRPR. Let*

$$f_{ij} \sim N\left(\frac{v_{ij}^- + v_{ij}^+}{2}, \left(\frac{v_{ij}^+ - v_{ij}^-}{6}\right)^2\right);$$

$$\mu_{ijk} = \frac{v_{ij}^- + v_{ij}^+ + v_{jk}^- + v_{jk}^+ - v_{ik}^- - v_{ik}^+ - 1}{2};$$

$$\sigma_{ijk} = \frac{\sqrt{(v_{ij}^+ - v_{ij}^-)^2 + (v_{jk}^+ - v_{jk}^-)^2 + (v_{ik}^+ - v_{ik}^-)^2}}{6};$$

and Φ be the cumulative distribution function of the standard normal distribution $N(0, 1)$. Then, the ACI of $\tilde{V}$ is expressed as follows:

$$ACI(\tilde{V}) = 1 - \frac{4}{n(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} ACI_{ijk}, \tag{11}$$

where

$$ACI_{ijk} = \begin{cases} \frac{2\sigma_{ijk}}{\sqrt{2\pi}}\left(e^{-\frac{\mu_{ijk}^2}{2\sigma_{ijk}^2}} - e^{\frac{-9}{2}}\right) + \mu_{ijk}\left(1 - 2\Phi\left(\frac{-\mu_{ijk}}{\sigma_{ijk}}\right)\right), & If \ -3 \leq \frac{-\mu_{ijk}}{\sigma_{ijk}} \leq 3 \\ \mu_{ijk}\left(\Phi(3) - \Phi(-3)\right), & If \ \frac{-\mu_{ijk}}{\sigma_{ijk}} \leq -3 \\ \mu_{ijk}\left(\Phi(-3) - \Phi(3)\right). & If \ \frac{-\mu_{ijk}}{\sigma_{ijk}} \geq 3 \end{cases}$$

Proof. Let us denote $x = f_{ij} + f_{jk} - f_{ik} - 0.5$. From Lemma 1, it is

$$x \sim N(\mu_{ijk}, \sigma_{ijk}^2)$$

with $\mu_{ijk} = \dfrac{v_{ij}^- + v_{ij}^+ + v_{jk}^- + v_{jk}^+ - v_{ik}^- - v_{ik}^+ - 1}{2}$ and $\sigma_{ijk} = \dfrac{\sqrt{(v_{ij}^+ - v_{ij}^-)^2 + (v_{jk}^+ - v_{jk}^-)^2 + (v_{ik}^+ - v_{ik}^-)^2}}{6}$.
According to Eq.(6), it is

$$ACI(\tilde{V}) = 1 - \frac{4}{n(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} E\big(|f_{ij} + f_{jk} - f_{ik} - 0.5|\big),$$

and therefore it would be

$$ACI(\tilde{V}) = 1 - \frac{4}{n(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} \frac{1}{\sigma_{ijk}\sqrt{2\pi}} \int_{\mu_{ijk}-3\sigma_{ijk}}^{\mu_{ijk}+3\sigma_{ijk}} |x| e^{-\frac{(x-\mu_{ijk})^2}{2\sigma_{ijk}^2}} dx$$

Let $y = \dfrac{x - \mu_{ijk}}{\sigma_{ijk}}$, then

$$ACI(\tilde{V}) = 1 - \frac{4}{n(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} \frac{1}{\sqrt{2\pi}} \int_{-3}^{3} |y\sigma_{ijk} + \mu_{ijk}| e^{-\frac{y^2}{2}} dy$$

We have:

$$|y\sigma_{ijk} + \mu_{ijk}| = \begin{cases} y\sigma_{ijk} + \mu_{ijk}, & y \geq \dfrac{-\mu_{ijk}}{\sigma_{ijk}} \\ -y\sigma_{ijk} - \mu_{ijk}, & y < \dfrac{-\mu_{ijk}}{\sigma_{ijk}} \end{cases}$$

- If $-3 \leq \dfrac{-\mu_{ijk}}{\sigma_{ijk}} \leq 3$:

$$\begin{aligned}
ACI_{ijk} = \frac{1}{\sqrt{2\pi}} \int_{-3}^{3} |y\sigma_{ijk} + \mu_{ijk}| e^{-\frac{y^2}{2}} dy = & \frac{1}{\sqrt{2\pi}} \int_{-3}^{\frac{-\mu_{ijk}}{\sigma_{ijk}}} (-y\sigma_{ijk} - \mu_{ijk}) e^{-\frac{y^2}{2}} dy \\
& + \frac{1}{\sqrt{2\pi}} \int_{\frac{-\mu_{ijk}}{\sigma_{ijk}}}^{3} (y\sigma_{ijk} + \mu_{ijk}) e^{-\frac{y^2}{2}} dy \\
= & \frac{\sigma_{ijk}}{\sqrt{2\pi}} \int_{-3}^{\frac{-\mu_{ijk}}{\sigma_{ijk}}} -y e^{-\frac{y^2}{2}} dy - \mu_{ijk} \frac{1}{\sqrt{2\pi}} \int_{-3}^{\frac{-\mu_{ijk}}{\sigma_{ijk}}} e^{-\frac{y^2}{2}} dy \\
& + \frac{-\sigma_{ijk}}{\sqrt{2\pi}} \int_{\frac{-\mu_{ijk}}{\sigma_{ijk}}}^{3} -y e^{-\frac{y^2}{2}} dy + \mu_{ijk} \frac{1}{\sqrt{2\pi}} \int_{\frac{-\mu_{ijk}}{\sigma_{ijk}}}^{3} e^{-\frac{y^2}{2}} dy \\
= & \frac{\sigma_{ijk}}{\sqrt{2\pi}} \left(e^{-\frac{y^2}{2}} \right)_{-3}^{\frac{-\mu_{ijk}}{\sigma_{ijk}}} - \mu_{ijk} \left(\Phi(\frac{-\mu_{ijk}}{\sigma_{ijk}}) - \Phi(-3) \right) \\
& + \frac{-\sigma_{ijk}}{\sqrt{2\pi}} \left(e^{-\frac{y^2}{2}} \right)_{\frac{-\mu_{ijk}}{\sigma_{ijk}}}^{3} + \mu_{ijk} \left(\Phi(3) - \Phi(\frac{-\mu_{ijk}}{\sigma_{ijk}}) \right) \\
= & \frac{2\sigma_{ijk}}{\sqrt{2\pi}} \left(e^{-\frac{\mu_{ijk}^2}{2\sigma_{ijk}^2}} - e^{\frac{-9}{2}} \right) + \mu_{ijk} \left(1 - 2\Phi(\frac{-\mu_{ijk}}{\sigma_{ijk}}) \right)
\end{aligned}$$

- If $\frac{-\mu_{ijk}}{\sigma_{ijk}} \leq -3$:

$$
\begin{aligned}
ACI_{ijk} = \frac{1}{\sqrt{2\pi}} \int_{-3}^{3} |y\sigma_{ijk} + \mu_{ijk}| e^{-\frac{y^2}{2}} dy =& \frac{1}{\sqrt{2\pi}} \int_{-3}^{3} (y\sigma_{ijk} + \mu_{ijk}) e^{-\frac{y^2}{2}} dy \\
=& \frac{-\sigma_{ijk}}{\sqrt{2\pi}} \int_{-3}^{3} -ye^{-\frac{y^2}{2}} dy + \mu_{ijk} \frac{1}{\sqrt{2\pi}} \int_{-3}^{3} e^{-\frac{y^2}{2}} dy \\
=& \frac{-\sigma_{ijk}}{\sqrt{2\pi}} \left(e^{-\frac{y^2}{2}} \right)_{-3}^{3} + \mu_{ijk} \left(\Phi(3) - \Phi(-3) \right) \\
=& \mu_{ijk} \left(\Phi(3) - \Phi(-3) \right)
\end{aligned}
$$

- If $\frac{-\mu_{ijk}}{\sigma_{ijk}} \geq 3$:

$$
\begin{aligned}
ACI_{ijk} = \frac{1}{\sqrt{2\pi}} \int_{-3}^{3} |y\sigma_{ijk} + \mu_{ijk}| e^{-\frac{y^2}{2}} dy =& \frac{1}{\sqrt{2\pi}} \int_{-3}^{3} (-y\sigma_{ijk} - \mu_{ijk}) e^{-\frac{y^2}{2}} dy \\
=& \frac{\sigma_{ijk}}{\sqrt{2\pi}} \int_{-3}^{3} -ye^{-\frac{y^2}{2}} dy - \mu_{ijk} \frac{1}{\sqrt{2\pi}} \int_{-3}^{3} e^{-\frac{y^2}{2}} dy \\
=& \frac{\sigma_{ijk}}{\sqrt{2\pi}} \left(e^{-\frac{y^2}{2}} \right)_{-3}^{3} - \mu_{ijk} \left(\Phi(3) - \Phi(-3) \right) \\
=& \mu_{ijk} \left(\Phi(-3) - \Phi(3) \right)
\end{aligned}
$$

Therefore, $ACI(\tilde{V}) = 1 - \frac{4}{n(n-1)(n-2)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} ACI_{ijk}$ and

$$
ACI_{ijk} = \begin{cases}
\frac{2\sigma_{ijk}}{\sqrt{2\pi}} \left(e^{-\frac{\mu_{ijk}^2}{2\sigma_{ijk}^2}} - e^{\frac{-9}{2}} \right) + \mu_{ijk} \left(1 - 2\Phi\left(\frac{-\mu_{ijk}}{\sigma_{ijk}} \right) \right), & If \ -3 \leq \frac{-\mu_{ijk}}{\sigma_{ijk}} \leq 3 \\
\mu_{ijk} \left(\Phi(3) - \Phi(-3) \right), & If \ \frac{-\mu_{ijk}}{\sigma_{ijk}} \leq -3 \\
\mu_{ijk} \left(\Phi(-3) - \Phi(3) \right). & If \ \frac{-\mu_{ijk}}{\sigma_{ijk}} \geq 3
\end{cases}
$$

This completes the proof. □

Corollary 1. *Let $\tilde{V}$ be an IVRPR. Then, (1) $ACI(\tilde{V}) \in [0,1]$, and (2) $ACI(\tilde{V}) = 1$ if and only if $\tilde{V}$ is an RPR and $CCI(\tilde{V}) = 1$.*

Proof. Because the average-case consistency index of $\tilde{V}$ is determined as the average consistency degree of all reciprocal preference relations associated to $\tilde{V}$, $ACI(\tilde{V}) \in [0,1]$. In the following, we prove (2).

- Sufficiency. Suppose that $\tilde{V}$ is an RPR and $CCI(\tilde{V}) = 1$. First, because $\tilde{V}$ is an RPR it is $CCI(\tilde{V}) = ACI(\tilde{V}) = CI(\tilde{V})$. Second, because $CCI(\tilde{V}) = 1$ it is also $ACI(\tilde{V}) = 1$.

- Necessity. Suppose that $ACI(\tilde{V}) = 1$. Using reductio ad absurdum, without loss of generality, let us assume that $v_{12}^{-} < v_{12}^{+}$. Let $F = (f_{ij})_{n\times n} \in N_{\tilde{V}}$ and $CI(F) = 1$. Then, we can construct a new RPR $F' = (f'_{ij})_{n\times n}$, where, $f'_{ij} = f_{ij}$ $(i, j \neq 1, 2)$ and $f'_{12} \in \{x | x \in [v_{12}^{-}, v_{12}^{+}], x \neq f_{12}\}$. Based on Eq. (2), $CI(F') \neq 1$. As a result, it would also be $ACI(\tilde{V}) \neq 1$, which contradicts the initial assumption. Thus, it has to be $v_{ij}^{-} = v_{ji}^{+}$ $\forall i, j$. Consequently, it would be $CCI(\tilde{V}) = ACI(\tilde{V}) = 1$.

This completes the proof. □

While corollary 1 provides the range of the ACI of an IVRPR, Theorem 1 provides the analytical procedure to compute the actual ACI value of an IVRPR $\tilde{V} = (\tilde{v}_{ij})_{n\times n}$. First, based on the preference values of $\tilde{V}$, we can get the values of μ_{ijk} and σ_{ijk} $(i, j, k = 1, 2, ..., n)$. Second, Eq. (11) and the cumulative distribution function of the standard normal distribution Φ allow the computation of the value of $ACI(\tilde{V})$.

3.2. Numerical examples for consistency measurement

This section provides numerical examples that illustrate the consistency measurement using CCI, BCI and ACI, respectively. Consider the following four IVRPRs:

$$\tilde{V}_1 = \begin{pmatrix} [0.5, 0.5] & [0.7, 1] & [0.1, 0.4] & [0.7, 1] \\ - & [0.5, 0.5] & [0.2, 1] & [0.2, 0.5] \\ - & - & [0.5, 0.5] & [0.4, 0.8] \\ - & - & - & [0.5, 0.5] \end{pmatrix}; \ \tilde{V}_2 = \begin{pmatrix} [0.5, 0.5] & [0.4, 1] & [0.2, 0.4] & [0, 0.5] \\ - & [0.5, 0.5] & [0.5, 1] & [0.6, 0.8] \\ - & - & [0.5, 0.5] & [0.6, 1] \\ - & - & - & [0.5, 0.5] \end{pmatrix}$$

$$\tilde{V}_3 = \begin{pmatrix} [0.5, 0.5] & [0, 1] & [0.4, 0.6] & [0.4, 0.6] \\ - & [0.5, 0.5] & [0.2, 0.8] & [0.3, 0.7] \\ - & - & [0.5, 0.5] & [0.3, 0.7] \\ - & - & - & [0.5, 0.5] \end{pmatrix}; \ \tilde{V}_4 = \begin{pmatrix} [0.5, 0.5] & [0, 0.6] & [0.1, 0.2] & [0.3, 0.4] \\ - & [0.5, 0.5] & [0, 0.8] & [0.3, 0.7] \\ - & - & [0.5, 0.5] & [0.3, 0.7] \\ - & - & - & [0.5, 0.5] \end{pmatrix}$$

Applying Eqs. (3) and (4) and Theorem 1 we derive the corresponding CCI, BCI and ACI values of the above four IVRPRs, which are given in Table 1 below:

Table 1: CCI, BCI and ACI values

	$\tilde{V}_1$	$\tilde{V}_2$	$\tilde{V}_3$	$\tilde{V}_4$
CCI	1	1	1	1
BCI	[0.63,0.77]	[0.5,0.83]	[0.7,0.7]	[0.67,0.77]
ACI	0.72	0.67	1	0.9545

3.3. Comparative Study

From Table 1, the following differences between the three different consistency measures of IVRPRs are highlighted:

(1) All four IVRPRS are consistent according to the classical consistency measure (CCI). However, according to the average-case consistency measure (ACI), it is noticed that two of the four IVRPRs do not have a very high consistency degree, while the other two do have a very high consistent degree.

(2) According to the boundary consistency index (BCI), consistency degrees for the four IVRPRS are always below 0.83 and above 0.5, which might not be considered very high.

These observations show that the average-case consistency (ACI) behaves differently to the classical consistency measure (CCI) and the boundary consistency measure (BCI). Looking in more detail the values of the IVRPRs, we observe the following further observations:

- From $\tilde{V}_1$, we observe the following:

  (i) It can be claimed that $x_1 \succ x_2$ because all possible preference values of alternative x_1 over alternative x_2 are above 0.7.

  (ii) It can be claimed that $x_3 \succ x_1$ because all possible preference values of alternative x_1 over alternative x_3 are below 0.4.

  (iii) It can be claimed that $x_1 \succ x_4$ because all possible preference values of alternative x_1 over alternative x_4 are above 0.7.

  (iv) It can be claimed that $x_4 \succ x_2$ because all possible preference values of alternative x_2 over alternative x_4 are below 0.5.

  (v) It is not clear that $x_3 \succ x_4$, however the possibility of being $x_3 \succ x_4$ is higher than that of being $x_4 \succ x_3$ as it deduces from the comparison of their respective interval-valued preferences $[0.4, 0.8]$ and $[0.2, 0.6]$, respectively. Indeed, given two interval numbers $\tilde{a}_1 = [a_1^-, a_1^+]$ and $\tilde{a}_2 = [a_2^-, a_2^+]$, the possibility degree (PD) up to which the ordering relation $\tilde{a}_1 \succ \tilde{a}_2$ is [33]:

$$P(\tilde{a}_1 \succ \tilde{a}_2) = \max\left\{1 - \max\left\{\frac{a_2^+ - a_1^-}{a_1^+ - a_1^- + a_2^+ - a_2^-}, 0\right\}, 0\right\} \tag{12}$$

  In the case of comparing alternatives with interval-valued reciprocal preferences, expression (12) can be used to conclude whether an alternative, x_i, is preferred to another one, x_j, by directly comparing the preference values $\tilde{v}_{ij} = [v_{ij}^-, v_{ij}^+]$ and $\tilde{v}_{ji} = [v_{ji}^-, v_{ji}^+]$:

  - $x_i \succ x_j \Leftrightarrow P(\tilde{v}_{ij} \succ \tilde{v}_{ji}) > 0.5$
  - $x_i \sim x_j \Leftrightarrow P(\tilde{v}_{ij} \succ \tilde{v}_{ji}) = 0.5$

Reciprocity of preferences, $[v_{ji}^-, v_{ji}^+] = [1 - v_{ij}^+, 1 - v_{ij}^-]$, reduces expression (12) to

$$P(\tilde{v}_{ij} \succ \tilde{v}_{ji}) = \max\left\{1 - \max\left\{\frac{0.5 - v_{ij}^-}{v_{ij}^+ - v_{ij}^-}, 0\right\}, 0\right\}$$

In our case, we have $\tilde{v}_{34} = [v_{34}^-, v_{34}^+] = [0.4, 0.8]$ and $\tilde{v}_{43} = [v_{43}^-, v_{43}^+] = [0.2, 0.6]$ and therefore:

$$P(\widetilde{v}_{34} \succ \widetilde{v}_{43}) = 0.75; \quad P(\widetilde{v}_{43} \succ \widetilde{v}_{34}) = 0.25$$

Putting all these consideration together for $\tilde{V}_1$, we have: $x_3 \succ x_1 \succ x_4 \succ x_2$. However, it is more possible that $x_2 \succ x_3$ rather than $x_3 \succ x_2$ because $P(\widetilde{v}_{23} \succ \widetilde{v}_{32}) = 5/8$; $P(\widetilde{v}_{32} \succ \widetilde{v}_{23}) = 3/8$, and therefore this inconsistency of preferences of $\tilde{V}_1$ is not accurately captured by the CCI measure.

- From $\tilde{V}_2$, we observe the following: $x_3 \succ x_1$, $x_4 \succ x_1$, $x_3 \succ x_4$, $x_2 \succ x_3$ so we can conclude that: $x_2 \succ x_3 \succ x_4 \succ x_1$. However, we have that $P(\widetilde{v}_{12} \succ \widetilde{v}_{21}) = 1/6$ and therefore it is more possible that $x_1 \succ x_2$. Again, CCI does not accurately represent the consistency of this IVRPR.
- From $\tilde{V}_3$, using possibility values it can be deduced that all four alternatives are equally preferred, which is not appropriately represented by BCI with a low consistency value of 0.7.
- From $\tilde{V}_4$, possibility values lead to: $x_4 \succ x_1$ and $x_3 \succ x_1$ with maximum possibility value, while at the same time we have $x_4 \sim x_3$. We also have $x_2 \succ x_1$ with possibility value $\frac{5}{6}$ and $x_3 \succ x_2$ with possibility value of $\frac{5}{8}$. Thus, preferences in this case are highly consistent with the ordering $x_3 \sim x_4 \succ x_2 \succ x_1$, but not completely consistent as there is also the relation $x_2 \sim x_4$. It is clear that the ACI value reflects this ordering relationships better than the BCI value (too low) and the CCI value (complete consistency), respectively.

In the following, we further analyse why the three different consistency measures for IVRPRs behave differently. According to Eq.(3), we have

$$CCI(\tilde{V}_i) = \max_{F \in N_{\tilde{V}_i}} CI(F), \quad i = 1, 2, 3, 4. \tag{13}$$

Solving the model described by expression (13) yields the associated RPR, $A_i \in N_{\tilde{V}_i}$, that satisfies

$$CCI(\tilde{V}_i) = CI(A_i), \quad i = 1, 2, 3, 4. \tag{14}$$

Similarly, we may define the worst consistency index (WCI) of the IVRPR, $\tilde{V}_i$, as follows:

$$WCI(\tilde{V}_i) = \min_{F \in N_{\tilde{V}_i}} CI(F), \quad i = 1, 2, 3, 4. \tag{15}$$

Solving the model described by expression (15) yields the associated RPR, $D_i \in N_{\tilde{V}_i}$, that satisfies

$$WCI(\tilde{V}_i) = CI(D_i), \quad i = 1, 2, 3, 4. \tag{16}$$

Table 2: Associated RPRs to IVRPRs for CCI, BCI and WCI

	A_i	B_i	C_i	D_i
$\tilde{V_1}$	$\begin{pmatrix} 0.5 & 0.7 & 0.4 & 0.7 \\ - & 0.5 & 0.2 & 0.5 \\ - & - & 0.5 & 0.8 \\ - & - & - & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 0.7 & 0.1 & 0.7 \\ - & 0.5 & 0.2 & 0.2 \\ - & - & 0.5 & 0.4 \\ - & - & - & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 1 & 0.4 & 1 \\ - & 0.5 & 1 & 0.5 \\ - & - & 0.5 & 0.8 \\ - & - & - & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 1 & 0.1 & 1 \\ - & 0.5 & 1 & 0.2 \\ - & - & 0.5 & 0.798 \\ - & - & - & 0.5 \end{pmatrix}$
$\tilde{V_2}$	$\begin{pmatrix} 0.5 & 0.4 & 0.4 & 0.5 \\ - & 0.5 & 0.5 & 0.6 \\ - & - & 0.5 & 0.6 \\ - & - & - & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 0.4 & 0.2 & 0 \\ - & 0.5 & 0.5 & 0.6 \\ - & - & 0.5 & 0.6 \\ - & - & - & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 1 & 0.4 & 0.5 \\ - & 0.5 & 1 & 0.8 \\ - & - & 0.5 & 1 \\ - & - & - & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 1 & 0.4 & 0 \\ - & 0.5 & 1 & 0.8 \\ - & - & 0.5 & 1 \\ - & - & - & 0.5 \end{pmatrix}$
$\tilde{V_3}$	$\begin{pmatrix} 0.5 & 0.4 & 0.6 & 0.6 \\ - & 0.5 & 0.7 & 0.7 \\ - & - & 0.5 & 0.5 \\ - & - & - & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 0 & 0.4 & 0.4 \\ - & 0.5 & 0.2 & 0.3 \\ - & - & 0.5 & 0.3 \\ - & - & - & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 1 & 0.6 & 0.6 \\ - & 0.5 & 0.8 & 0.7 \\ - & - & 0.5 & 0.7 \\ - & - & - & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 1 & 0.6 & 0.4 \\ - & 0.5 & 0.8 & 0.7 \\ - & - & 0.5 & 0.7 \\ - & - & - & 0.5 \end{pmatrix}$
$\tilde{V_4}$	$\begin{pmatrix} 0.5 & 0.2 & 0.2 & 0.4 \\ - & 0.5 & 0.5 & 0.7 \\ - & - & 0.5 & 0.7 \\ - & - & - & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 0 & 0.1 & 0.3 \\ - & 0.5 & 0 & 0.3 \\ - & - & 0.5 & 0.3 \\ - & - & - & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 0.6 & 0.2 & 0.4 \\ - & 0.5 & 0.8 & 0.7 \\ - & - & 0.5 & 0.7 \\ - & - & - & 0.5 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 0 & 0.1 & 0.4 \\ - & 0.5 & 0 & 0.3 \\ - & - & 0.5 & 0.3 \\ - & - & - & 0.5 \end{pmatrix}$

Based on Eq.(1), we can easily get the boundary RPRs B_i and C_i associated to $\tilde{V_i}$ $(i = 1, 2, 3, 4)$, and

$$BCI(\tilde{V_i}) = \Big[\min\big\{CI(B_i), CI(C_i)\big\}, \max\big\{CI(B_i), CI(C_i)\big\}\Big]. \tag{17}$$

Tables 2 and 3 provide the above associated RPRs, A_i, B_i, C_i, D_i, and their corresponding consistency values, respectively.

Clearly, the data in Tables 1-3 is in line with Eqs. (14), (16) and (17). According to Eqs. (13) and (14), the CCI of the IVRPR $\tilde{V_i}$ is determined by its associated RPR with highest consistency degree, A_i. Meanwhile, according to Eqs. (15) and (16), the WCI of the IVRPR $\tilde{V_i}$ is determined by its associated RPR with worst consistency degree, D_i. In other words, the CCI and WCI values are the upper and lower bounds for the consistency degree of IVRPRs, respectively. Moreover, based on Eq.(17), the BCI of $\tilde{V_i}$ is determined by its two associated boundary RPRs, B_i and C_i, whose consistency values will obviously be bounded by the WCI and CCI values of $\tilde{V_i}$, i.e. $BCI(\tilde{V_i}) \in [WCI(\tilde{V_i}), CCI(\tilde{V_i})]$. However, as Wang shown recently in [29], both B_i and C_i cannot provide reliable information to measure the consistency degree of $\tilde{V_i}$.

Different from the classical consistency measure and the boundary consistency measure, the ACI

Table 3: Consistency degrees of Associated RPRs

	A_i	B_i	C_i	D_i
$i = 1$	1	0.77	0.63	0.43
$i = 2$	1	0.83	0.5	0.33
$i = 3$	1	0.7	0.7	0.63
$i = 4$	1	0.67	0.77	0.63

of an IVRPR is determined using all the IVRPR associated RPRs' consistency values. Obviously, in this case the following also holds:

$$ACI(\tilde{V}_i) \in [WCI(\tilde{V}_i), CCI(\tilde{V}_i)]. \tag{18}$$

In summary, the CCI, the WCI and the ACI provide the upper bound, lower bound and average consistency degree of IVRPRs, respectively, and each one complements the other, making their combined use comprehensively reflect the consistency status of IVRPRs.

4. Average-case consistency improving method

For RPRs of unacceptable consistency, consistency improving methods [8–10, 36, 37] have been developed. In this section, an average-case consistency improving method with the aim of obtaining a modified IVRPR with a required ACI is developed.

4.1. A method to improve ACI

The basic idea of the proposed average-case consistency improving method is based on the concept of adjusted IVRPR of a given IVRPR and their relationship regarding their CCI and WCI values as per the following definition and results:

***Definition* 5 (Adjusted IVRPR).** Let $\tilde{V} = ([v_{ij}^-, v_{ij}^+])_{n\times n}$ be an IVRPR. $A = (a_{ij})_{n\times n}$ the RPR associated to $\widetilde{V}$ with best consistency degree, i.e. $CI(A) = CCI(\tilde{V})$; and $D = (d_{ij})_{n\times n}$ the RPR associated to $\widetilde{V}$ with worst consistency degree, i.e. $CI(D) = WCI(\tilde{V})$. The preference relation $\widetilde{V}' = ([v_{ij}'^-, v_{ij}'^+])_{n\times n}$, constructed according to the following three rules

R1: If $a_{ij} < d_{ij}$, then let $v_{ij}'^- = v_{ij}^-$ and $v_{ij}'^+ \in [a_{ij}, d_{ij})$.

R2: If $a_{ij} > d_{ij}$, then let $v_{ij}'^- \in (d_{ij}, a_{ij}]$ and $v_{ij}'^+ = v_{ij}^+$.

R3: If $a_{ij} = d_{ij}$, then let $v_{ij}'^- = v_{ij}^-$ and $v_{ij}'^+ = v_{ij}^+$.

is called the adjusted IVRPR associated to $\widetilde{V}$.

Given an IVRPR with specific WCI value, the following result allows to improve the WCI value while, simultaneously, preserving the IVRPR CCI value by computing the corresponding associated adjusted IVRPR.

***Lemma* 2.** *Let $\tilde{V}$ be an IVRPR and $\tilde{V}'$ be its adjusted IVRPR. Then it is $CCI(\widetilde{V}') = CCI(\widetilde{V})$, and $WCI(\widetilde{V}') \geq WCI(\widetilde{V})$.*

Proof. According to R1-R3, we have that $[v'^{-}_{ij}, v'^{+}_{ij}] \subset [v^{-}_{ij}, v^{+}_{ij}]$ and therefore any RPR associated to $\widetilde{V}'$ is also an RPR associated to $\widetilde{V}$. Also, R1-R3 imply that $a_{ij} \in [v'^{-}_{ij}, v'^{+}_{ij}]$ in the adjustment process, and consequently it is: $CCI(\widetilde{V}') = CCI(\widetilde{V}) = CI(A)$. On the other hand, R1-R3 imply that $d_{ij} \notin [v'^{-}_{ij}, v'^{+}_{ij}]$, and it can be concluded that $WCI(\widetilde{V}') \geq WCI(\widetilde{V}) = CI(D)$. This completes the proof. $\square$

A direct consequence of Lemma 2 is that an IVRPR $\widetilde{V}$ and its adjusted IVRPR $\widetilde{V}'$ have the same CCI value.

***Corollary* 2.** *Let $\tilde{V}$ be an IVRPR and $\widetilde{V}'$ its adjusted IVRPR. If $A = (a_{ij})_{n\times n}$ is the RPR associated to $\widetilde{V}$ with best consistency degree, i.e. $CI(A) = CCI(\tilde{V})$, then $CI(A) = CCI(\tilde{V}')$.*

According to Eq.(18), ACI is located between WCI and CCI, and therefore it is feasible to develop an average-case consistency improving method to improve the ACI of IVRPRs by increasing the value of WCI. To achieve this, a linear programming based method to improve the CCI of IVRPRs is presented.

Linear programming based method to improve the CCI of IVRPRs. Let $\tilde{V} = ([v^{-}_{ij}, v^{+}_{ij}])_{n\times n}$ be an IVRPR. The main aim when improving the CCI of $\widetilde{V}$ is to find a suitable IVRPR, $\widetilde{V}^{*} = ([v^{*-}_{ij}, v^{*+}_{ij}])$, with $CCI(\widetilde{V}^{*}) = 1$. To preserve the information in $\widetilde{V}$ as much as possible, $\widetilde{V}^{*}$ is chosen as the IVRPR closest to $\widetilde{V}$. When the distance between two IVRPRs is computed using the Manhattan distance [5, 6], we have that $\widetilde{V}^{*}$ is the solution to the following linear programming problem:

$$(LP-1)\begin{cases} \min \dfrac{1}{n(n-1)} \displaystyle\sum_{i=1}^{n} \sum_{j=i+1}^{n} (|v^{+}_{ij} - v^{*+}_{ij}| + |v^{-}_{ij} - v^{*-}_{ij}|) \\ s.t.\;\; CCI(\widetilde{V}^{*}) = 1 \end{cases}$$

The below algorithm 1 provides a formal description of the proposed average-case consistency improving method.

Input: The IVRPR $\widetilde{V} = (\widetilde{v}_{ij})_{n\times n}$, and the average consistency threshold $\overline{ACI}$.

Output: The adjusted IVRPR $\widetilde{V}' = (\widetilde{v}'_{ij})_{n\times n}$, and $ACI(\widetilde{V}')$.

Step1. Let $t = 0$, and $\widetilde{V}^t = \widetilde{V}$.

Step 2. **if** $CCI(\widetilde{V}^t) < 1$ **then**

| apply $(LP-1)$ method to obtain new IVRPR $\widetilde{V}^{t*}$, such that $CCI(\widetilde{V}^{t*}) = 1$.

| Let $\widetilde{V}^t = \widetilde{V}^{t*}$ and go to **Step 3.**

else

| go to **Step 3.**

end if

Step 3. Apply Theorem 1 to calculate $ACI(\widetilde{V}^t)$.

if $ACI(\widetilde{V}^t) < \overline{ACI}$ **then**

| go to **Step 4.**

else

| go to **Step 5.**

end if

Step 4. Compute associated RPRs to $\widetilde{V}^t$, $A^t = (a^t_{ij})_{n\times n}$ and $D^t = (d^t_{ij})_{n\times n}$, such that: $CI(A^t) = CCI(\tilde{V}^t)$ and $CI(D^t) = WCI(\tilde{V}^t)$. Apply R1-R3 from Definition 5 to compute the adjusted IVRPR, $\widetilde{V}^{t+1}$, associated to $\widetilde{V}^t$. Let $t = t+1$, and go to **Step 3.**

Step 5. $\widetilde{V}' = \widetilde{V}^t$.

Algorithm 1: IVRPR average-case consistency improving method.

The following result proves that when the average consistency threshold $\overline{ACI}$ increases towards its maximum value 1 then the average-case consistency measure of the adjusted IVRPR $\widetilde{V}' = (\widetilde{v}'_{ij})_{n\times n}$ derived from Algorithm 1, $ACI(\widetilde{V}')$, also increases towards its maximum value 1.

***Theorem* 2.** *Let $\tilde{V}$ be an IVRPR, and $\widetilde{V}'$ the adjusted IVRPR derived from Algorithm 1 with an average consistency threshold $\overline{ACI}$. Then:*

$$\lim_{\overline{ACI}\to 1} ACI(\widetilde{V}') = 1.$$

Proof. Based on Definition 5, we have that d^{t+1}_{ij} values obtained from Rule 1 are lower than the corresponding value d^t_{ij} and greater than a_{ij}, while d^{t+1}_{ij} values obtained from Rule 2 are greater than the corresponding value d^t_{ij} and lower than a_{ij}, respectively. Values d^{t+1}_{ij} obtained from Rule 3 are equal to the corresponding value d^t_{ij} and also equal to a_{ij}. Thus the RPR D^{t+1} is closer to A than RPR D^t. Furthermore, each element of D^t can be classed as being in a strictly monotonic sequence bounded by the corresponding element of A, and therefore when Lemma 2 is repeatedly applied then we have that the sequence of RPRs $\{D^t; t \in \mathbb{N}\}$ converges towards A. Therefore, when $\overline{ACI} \to 1$ we have that $D^t \to A$, and because $ACI(\tilde{V}) \in [WCI(\tilde{V}), CCI(\tilde{V})]$ it is

$$\lim_{\overline{ACI}\to 1} ACI(\widetilde{V}') = 1.$$

This completes the proof. $\square$

Theorem 2 guarantees that the proposed average-case consistency improving method can transform any IVRPR into one with a required ACI.

The design of the consistency improving method is a classical topic in decision making with preference relations. Generally, the adjusted values should only be considered as a decision aid which decision makers use as a reference to modify his preference values. The proposed average-case consistency improving method follows this research line, and both (LP-1) and R1-R3 should be used as a reference for decision makers to improve the consistency level of IVRPRs.

4.2. Numerical example for consistency improvement

Next, we use the IVRPR $\widetilde{V}_1$ presented in Section 3.2 as an example to illustrate the use of the average-case consistency improving method. Without loss of generality, in this example, it is set $\overline{ACI} = 0.9$.

Algorithm 1 – Iteration 1. Step 1. Let $\widetilde{V}_1^0 = \widetilde{V}_1$. According to Table 3, we have $CCI(\widetilde{V}_1^0) = CI(A_1^0) = 1$, with A_1^0 given in Table 2

$$A_1^0 = \begin{pmatrix} 0.5 & 0.7 & 0.4 & 0.7 \\ - & 0.5 & 0.2 & 0.5 \\ - & - & 0.5 & 0.8 \\ - & - & - & 0.5 \end{pmatrix}.$$

Go to **Step 3**. From Table 1, we have $ACI(\widetilde{V}_1^0) = 0.72$, which is lower than $\overline{ACI}$. Go to **Step 4**. Solving Eq.(15), the associated RPR to $\widetilde{V}_1^0$ with worst consistency degree is

$$D_1^0 = \begin{pmatrix} 0.5 & 1 & 0.1 & 1 \\ - & 0.5 & 1 & 0.2 \\ - & - & 0.5 & 0.798 \\ - & - & - & 0.5 \end{pmatrix}.$$

According to R1-R3 from Definition 5, the directions to improve the ACI of $\widetilde{V}_1^0$ are:

(i) Increase the value of $v_{13}^{0,-}$, $v_{24}^{0,-}$ and $v_{34}^{0,-}$:

$$v_{13}^{-,1} \in (0.1, 0.4];\; v_{24}^{-,1} \in (0.2, 0.5];\; v_{34}^{-,1} \in (0.798, 0.8].$$

(ii) Decrease the value of $v_{12}^{0,+}$, $v_{14}^{0,+}$ and $v_{23}^{0,+}$:

$$v_{12}^{+,1} \in [0.7, 1);\; v_{14}^{+,1} \in [0.7, 1);\; v_{23}^{+,1} \in [0.2, 1).$$

Following the above suggestions and assuming, without loss of generality, that the new (adjusted) IVRPR $\widetilde{V}_1^1 = (\widetilde{v}_{ij}^1)$ given is

$$\widetilde{V}_1^1 = \begin{pmatrix} [0.5, 0.5] & [0.7, 0.8] & [0.2, 0.4] & [0.7, 0.9] \\ - & [0.5, 0.5] & [0.2, 0.8] & [0.3, 0.5] \\ - & - & [0.5, 0.5] & [0.799, 0.8] \\ - & - & - & [0.5, 0.5] \end{pmatrix}.$$

From Corollary 2, it is $A_1^1 = A_1^0$ and therefore $CCI(\widetilde{V}_1^1) = CI(A_1^1) = CI(A_1^0) = 1$ and therefore go to **Step 3** for a new iteration of Algorithm 1.

Algorithm 1 – Iteration 2. **Step 3**. Eq.(11) results in $ACI(\widetilde{V}_1^1) = 0.8$, which is still below the threshold value $\overline{ACI}$. Go to **Step 4**. Solving Eq.(15), the associated RPR to $\widetilde{V}_1^1$ with worst consistency degree is

$$D_1^1 = \begin{pmatrix} 0.5 & 0.798 & 0.2 & 0.9 \\ - & 0.5 & 0.8 & 0.3 \\ - & - & 0.5 & 0.799 \\ - & - & - & 0.5 \end{pmatrix}.$$

According to R1-R3 from Definition 5, the directions to improve the ACI of $\widetilde{V}_1^1$ are:

(i) Increase the value of $v_{13}^{-,1}$, $v_{24}^{-,1}$ and $v_{34}^{-,1}$:

$$v_{13}^{-,2} \in (0.2, 0.4];\; v_{24}^{-,2} \in (0.3, 0.5];\; v_{34}^{-,2} \in (0.799, 0.8].$$

(ii) Decrease the value of $v_{12}^{+,1}$, $v_{14}^{+,1}$ and $v_{23}^{+,1}$:

$$v_{12}^{+,2} \in [0.7, 0.798);\; v_{14}^{+,2} \in [0.7, 0.9);\; v_{23}^{+,2} \in [0.2, 0.8).$$

Following the above suggestions and assuming, without loss of generality, that the new (adjusted) IVRPR $\widetilde{V}_1^2 = (\widetilde{v}_{ij}^2)$ is

$$\widetilde{V}_1^2 = \begin{pmatrix} [0.5, 0.5] & [0.7, 0.75] & [0.3, 0.4] & [0.7, 0.8] \\ - & [0.5, 0.5] & [0.2, 0.5] & [0.4, 0.5] \\ - & - & [0.5, 0.5] & [0.8, 0.8] \\ - & - & - & [0.5, 0.5] \end{pmatrix}.$$

A new iteration of Algorithm 1 is carried out.

Algorithm 1 – Iteration 3. **Step 3**. Because $ACI(\widetilde{V}_1^2) = 0.9$, the threshold value $\overline{ACI}$ has been reached. Go to **Step 5**, which ends the algorithm and returns as outputs:

$$\widetilde{V}' = \widetilde{V}_1^2;\quad ACI(\widetilde{V}') = 0.9.$$

4.3. Simulation experiments

In this subsection, we explore the average-case consistency improving method by means of simulation experiments. Let $\widetilde{V}$, $\widetilde{V}'$, A and D as per Definition 5. In order to show the process to improve the values CCI, WCI and ACI of $\widetilde{V}$ when applying the presented average-case consistency improving method, a parameter α $(0 < \alpha < 1)$ is introduced to automatically revise the preference values in $\widetilde{V}$ to derive its adjusted IVRPR $\widetilde{V}'$. To do so, R1-R3 in Definition 5 are re-defined as follows:

R1': If $a_{ij} < d_{ij}$, then let $v'^{-}_{ij} = v^{-}_{ij}$ and $v'^{+}_{ij} = max\{a_{ij}, d_{ij} - \alpha\}$.

R2': If $a_{ij} > d_{ij}$, then let $v'^{-}_{ij} = min\{d_{ij} + \alpha, a_{ij}\}$ and $v'^{+}_{ij} = v^{+}_{ij}$.

R3': If $a_{ij} = d_{ij}$, then let $v'^{-}_{ij} = v^{-}_{ij}$ and $v'^{+}_{ij} = v^{+}_{ij}$.

The larger the parameter α value is, the larger the adjustment amount will be in R1'-R3'. In Algorithm 1, we replace R1-R3 with R1'-R3', respectively, and then obtain an automatic version of Algorithm 1, that we will refer to as Algorithm 2. Next, we set different α values, and run Algorithm 2 to improve the consistency indexes (CCI, WCI and ACI) of IVRPRs. Because $ACI(\widetilde{V}_3) = 1$ and $ACI(\widetilde{V}_4) = 0.9545$ are already quite high, we replace them with two new IVRPRs, $\widetilde{V}_5$ and $\widetilde{V}_6$, taken from [35] and [28], respectively:

$$\widetilde{V}_5 = \begin{pmatrix} [0.5, 0.5] & [0.3, 0.4] & [0.5, 0.7] & [0.4, 0.5] \\ - & [0.5, 0.5] & [0.6, 0.8] & [0.2, 0.6] \\ - & - & [0.5, 0.5] & [0.4, 0.8] \\ - & - & - & [0.5, 0.5] \end{pmatrix}; \widetilde{V}_6 = \begin{pmatrix} [0.5, 0.5] & [0.75, 0.85] & [0.65, 0.75] & [0.35, 0.45] \\ - & [0.5, 0.5] & [0.5, 0.65] & [0.5, 0.65] \\ - & - & [0.5, 0.5] & [0.62, 0.75] \\ - & - & - & [0.5, 0.5] \end{pmatrix}.$$

Algorithm 2 is applied to $\widetilde{V}_1$, $\widetilde{V}_2$, $\widetilde{V}_5$ and $\widetilde{V}_6$ with three different α values of 0.05, 0.1 and 0.15. The improving process of the consistency indexes for each IVRPR is illustrated in Figures 1-4, respectively. The following observations can be drawn:

(1) Notice that CCI if not 1 at iteration 1 of Algorithm 2, then it is set to 1 from iteration 2 and remains as such thereafter.

(2) Both the ACI and WCI values of $\widetilde{V}_i$ $(i = 1, 2, 5, 6)$ increase in each iteration. This was already proved theoretically in Lemma 2 and Theorem 2, respectively. This implies that both the ACI and WCI can be improved by using the average-case consistency improving method, which further justifies the feasibility of our proposal.

(3) It will take less iterations to reach an established consistency index the larger the α value is. Indeed, the larger the value of α the closer the RPR D is to the RPR A, and consequently the ACI value will be pushed closer to the CCI value, which is set to its maximum value of 1 in the first iteration of the proposed algorithm.

5. Conclusion

In this paper, the average-case consistency measure of IVRPRs has been proposed, analysed and compared against the two existing consistency measures of IVRPR: the classical consistency measure and the boundary consistency measure. The underlying idea of the average consistency measure consist in measuring the consistency degree of the IVRPR using the average of the consistency degrees of all its associated RPRs. The internal mechanisms of the different consistency measures have been analysed, and the combined use of the classical consistency measure, the worst consistency measure and the average consistency measure are proposed to comprehensively reflect the consistency status of IVRPRs. Furthermore, an average-case consistency improving method aimed to obtain a modified IVRPR with a required ACI is proposed, theoretically justified and corroborated via an experiment simulation. This consistency improvement method relies on the concept of adjusted IVRPR associated to a given IVRPR and on the application of linear programming based method to improve the classical consistency index of an IVRPR.

In future, the following issues need attention:

(1) The consistency measure has been used as a driver to estimate missing information in incomplete RPRs [15]. In future, it will be interesting to study a method based on the ACI to estimate the missing values of incomplete IVRPRs, and to compare it with existing approaches based on the CCI, as the ones reported in [25, 26, 31, 32].

(2) The proposed average-case consistency measure of IVRPRs is based on the additive transitivity. Since the multiplicative transitivity is an alternative approach to measure consistency of RPRs [4], it will be interesting to study the mathematical properties of a corresponding average multiplicative consistency of IVRPRs.

(3) Establishing the consistency threshold $\overline{CI}$ for RPRs is a very challenging task, and is still open. It will be necessary to provide a systematic investigation to establish the thresholds for the consistency indexes CI, CCI, WCI and ACI.

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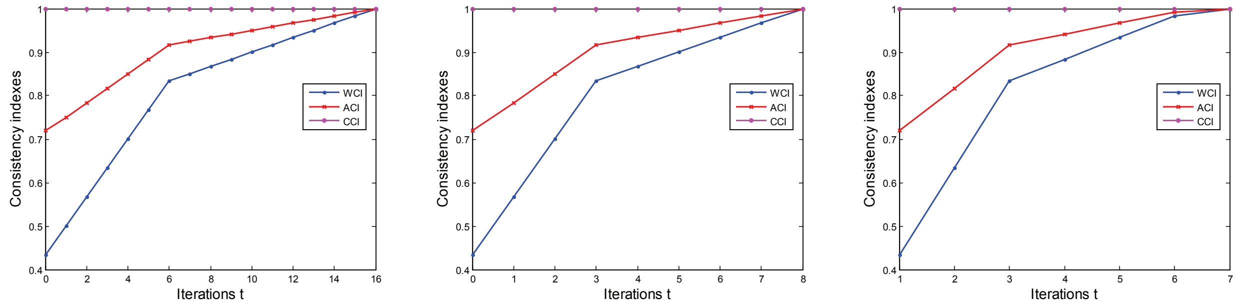

(a) α=0.05 (b) α=0.1 (c) α=0.15

Figure 1. The process to improve the consistency indexes of $\widetilde{V}_1$ in Algorithm 2

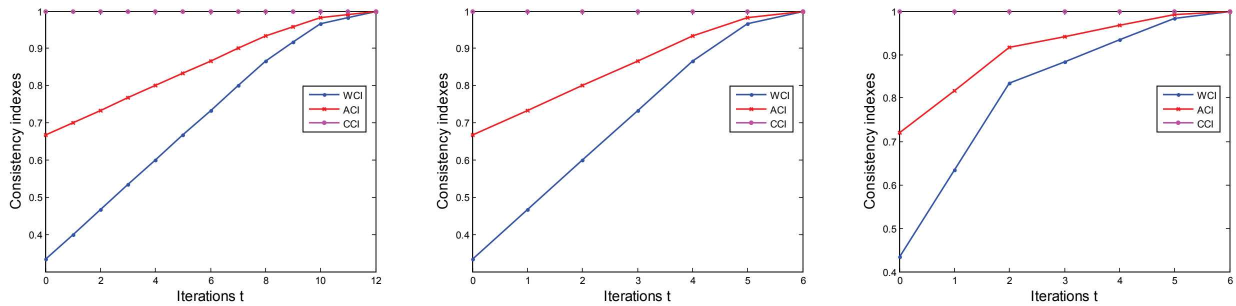

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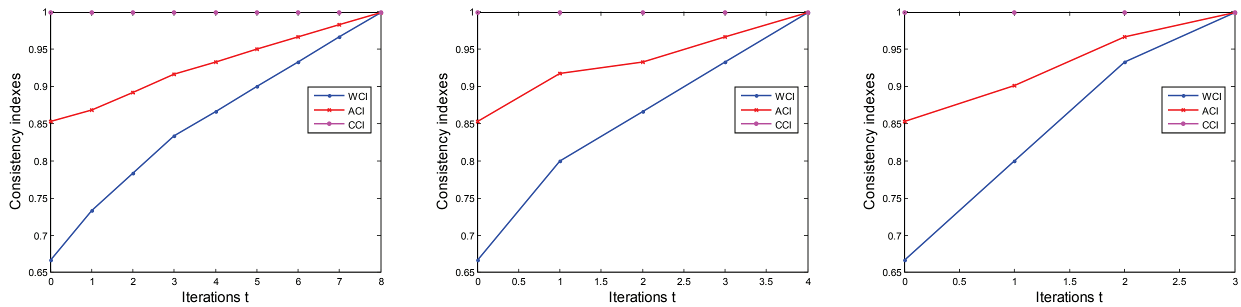
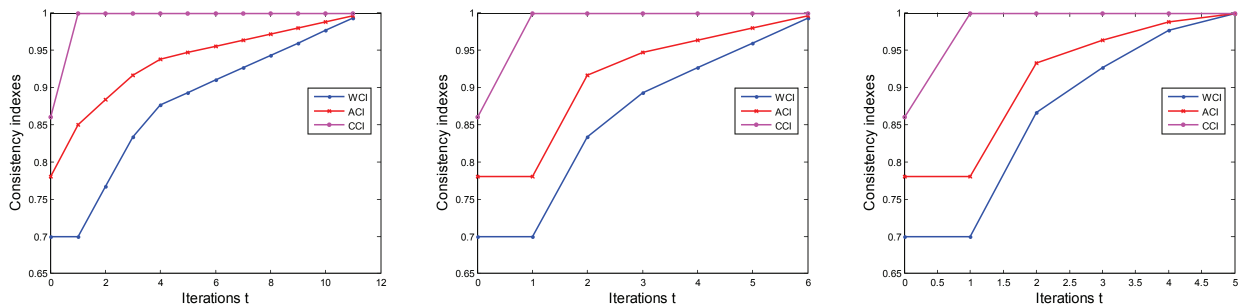
Figure 2. The process to improve the consistency indexes of $\widetilde{V}_2$ in Algorithm 2



(a) α=0.05 (b) α=0.1 (c) α=0.15

Figure 3. The process to improve the consistency indexes of $\widetilde{V}_5$ in Algorithm 2



(a) α=0.05 (b) α=0.1 (c) α=0.15

Figure 4. The process to improve the consistency indexes of $\widetilde{V}_6$ in Algorithm 2