Linguistic Multi-Criteria Decision-Making Model with Output Variable Expressive Richness

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Abstract. In general, traditional decision-making models are based on methods that perform calculations on quantitative measures. These methods are usually applied to assess possible solutions to a problem, resulting in a ranking of alternatives. However, when it comes to making decisions about qualitative measures –such as service quality–, the quantitative assessment is a bit difficult to interpret. Therefore, taking into account the maturity of the linguistic assessment models, this paper puts forth a new solution proposal. It is a decision-making model that uses linguistic labels –represented with the 2-tuple notation– and a variable expressive richness when providing output results. This solution allows expressing results in a manner closer to the human cognitive system. To achieve this goal, a mechanism has been implemented for measuring the distance among the aggregate ratings, providing the decision-maker with a fast and intuitive answer. The proposal is illustrated with an application example based on the TOPSIS model, using linguistic labels throughout the entire process.

Keywords: multi-criteria decision-making, linguistic labels, variable expressive richness, 2-tuple representation, linguistic TOPSIS model.

1. Introduction

Multi-criteria decision-making (MCDM) is present in the day-to-day life of companies (Figueira, Greco & Ehrgott, 2005; Zavadskas & Turskis, 2011). It is a process through which the best solution to a problem is sought among a set of possible solutions. There are several MCDM models based on the so-called compensatory methods including aspects related to costs and benefits. Some good examples among many others are the TOPSIS method (Behzadian, Otaghsara, Yazdani & Ignatius, 2012), a performance-ranking method based on the resemblance to the ideal solution, the AHP method (Saaty, 2008), with its limitations in terms of the number of alternatives it can analyze, and the QFD model (Chan & Wu, 2002), for decision-making on the quality of products and services. However, these solutions have been developed for assessing problems that involve quantitative variables, that is, for cases where the dimensions or criteria used are expressed numerically.

Some studies –like the ones conducted by Y. J. Wang and H. S. Lee (2007), T. C. Wang and H. D. Lee (2009), Sun (2010), Sipahi and Timor (2010), and Low and Lin (2013), among others– propose alternative solutions to traditional information processing and have been used in different decision-making (DM) areas, such as fuzzy models, determination of weights, data mining, etc. However, it is necessary to seek solutions to DM problems from a closer perspective to human thought and expression. Since natural language is the most

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widely used communication mechanism by humans, it would be useful to develop a method closer to natural language, which expresses results in a more understandable way for decision-makers and thus makes the DM process easier. In general, linguistic models are based on the use of descriptive semantics related to the particular topic at hand. Here it should be noted that some methods use variable expressive richness initially (Behzadian et al., 2012), while some papers approach this problem from a multi-granularity perspective (Herrera, Herrera-Viedma & Martínez, 2000; Massanet, Riera, Torrens & Herrera-Viedma, 2014; Morente-Molinera, Al-Hmouz, Morfeq, Balamash & Herrera-Viedma, 2016; Morente-Molinera, Pérez, Ureña & Herrera-Viedma, 2015), thus allowing experts with different levels of expertise to express their assessments in a more flexible manner. Other models provide a solution combining the use of output linguistic labels with input quantitative information (Herrera & Martínez, 2000a). Some studies also consider using the 2-tuple representation throughout the entire process (Carrasco, Muñoz-Leiva & Hornos, 2013; Carrasco, Villar, Hornos & Herrera-Viedma, 2011, 2012; Cid-López, Hornos, Carrasco & Herrera-Viedma, 2015, 2016; Dong & Herrera-Viedma, 2015; Tejeda-Lorente, Porcel, Peis, Sanz & Herrera-Viedma, 2014), thus ensuring that no information will be lost in the process. The papers mentioned in this section are just some of the examples proposed for MCDM from a wide variety of solutions found in literature.

This paper puts forward a new MCDM model based on human thinking, hence introducing an alternative solution to the representation of results according to their complexity and using a new mechanism called *Variable Expressive Richness* (VER).

To better understand this proposal, let us suppose we have a DM problem which, for instance, uses a specific number of linguistic labels to express final results, then it is possible that different results are expressed with the same label, which finally makes the decision-maker's work more complicated. By using the proposed VER mechanism, the number of output labels does not need to be predefined, since it will be automatically adjusted to the ideal set of labels expressing the corresponding results. In other words, we propose using an expressive richness that will vary according to the final results –provided by any base system– for the problem at hand. In order to achieve this, the distance between the previously sorted final results needs to be calculated. The lowest value obtained (minimum distance) will determine the most appropriate set of linguistic labels for expressing the corresponding results. Label sets make up a multi-granular system containing different levels of label sets, which will be reflected in the variety of answers generated. To illustrate this proposal, the implementation algorithm of the Linguistic TOPSIS (LTOPSIS) model will be used as a base model, with the proposed VER module connected to its output in order to better express the results obtained.

In our proposal, the linguistic labels that make up the fuzzy sets used are represented using a triangular membership function, which generates a set of odd labels distributed symmetrically in a balanced interval around a central label. Although there are other ways of representing fuzzy sets, this is the one we have chosen to do it.

Miller (1956) suggested using 7 (plus or minus 2) categories (i.e. options) to *initially assess* the different criteria considered for the alternatives, since this task is carried out by users or experts. Following such suggestion, we use few (usually, 5) linguistic labels to perform such task. However, there are cases (such as the ones presented in Section 4) where the *final results* for the different alternatives may coincide, being necessary to apply some mechanism that allows differentiating them. It is in such cases where the application of our proposal will help the decision-maker to choose more easily the best solution alternative.

The rest of the article is structured as follows: Section 2 sets out the materials and methodology necessary to explain how the basic elements of our proposal work; Section 3 provides a detailed presentation of our proposal, explaining both the base model used and the changes implemented to obtain the new model; Section 4 describes a case study, analyses the results obtained and presents additional examples of use of the VER module. Finally, Section 5 displays the conclusions and future work.

2. Materials and methodology

This section puts forward the theoretical foundations used in our proposal by shortly describing them.

2.1 Linguistic variables

It is very common for the decision-maker to encounter difficulties in defining the importance of a set of criteria and/or the appropriateness of an alternative for a given set of criteria, especially if she/he uses a numerical evaluation method. Hence the importance of providing the appropriate tools that will make the decision-maker's work easier. In this sense, we are sure that using the widely known linguistic variables would hugely facilitate this task. Since Zadeh introduced the 'fuzzy set' and 'linguistic variable' concepts (Zadeh, 1975, 1983, 1996), the use and popularity of fuzzy logics has been outstanding. In this case, we are interested in the role linguistic variables play as an ordinal scale, as well as in their application to the MCDM.

The concept of linguistic variable (Zadeh, 1975) can be understood as a variable that takes values in a context of words or sentences expressed in natural language. For instance, the quality of a service from the user's perspective can be considered a linguistic variable if its values are expressed linguistically (e.g. *Extremely Poor*, *Very Poor*, *Poor*, *Fair*, *Good*, *Very Good*, *Excellent*) instead of numerically (e.g. 0,...,15,…,25,…,50,...,80,…,100). Therefore, linguistic variables can be defined as an ordered set of linguistic terms or labels, $S = \{s_i \mid i = 1, \dots, n\}$, where $s_i \prec s_j \Leftrightarrow i < j$.

Definition 1: According to Zadeh, a linguistic variable is characterized by a quintuple with the following structure:

$$\{X; T(X); U; G; M\}$$

where:

- X is the name of the variable,
- $T(X)$ is the set of linguistic terms (or labels) defined or contained in it,
- U is the universe of discourse of the variable,
- G is the syntactic rule to generate the elements of the $T(X)$ set, and
- M is the semantic rule that assigns a meaning to each element of the $T(X)$ set.

The amount of elements in $T(X)$ could vary depending on the expressive richness necessary for each case, or, in other words, depending on the context of the DM process. Miller (1956) established that the number of labels can be determined according to the context. Having sufficient linguistic label sets with different numbers of labels allows enriching the expression of results and makes them easier to understand.

These different label sets can be expressed as S^t, where $t \in \{1, \dots, q\}$ and q is the number of levels in the linguistic hierarchy employed. Therefore, each of the different labels can be represented by Equation (1), where $n(t)$ is the granularity or number of labels at the level t:

$$s_i^t \in \{s_1^t, \dots, s_{n(t)}^t\}, \forall t \in \{1, .., q\}, \forall i \in \{1, .., n(t)\} \qquad (1)$$

Table 1 shows examples of variability of the elements in $T(X)$ for one linguistic variable, depending on the context or the expressive richness needed.

Table 1. Different sets of (3, 5, 9 and 17) labels for the same variable.

<table>
<tr><th>$t = 1$
$n(1) = 3$
$S^1 = \{s_1^1, \dots, s_3^1\}$</th><th>$t = 2$
$n(2) = 5$
$S^2 = \{s_1^2, \dots, s_5^2\}$</th><th>$t = 3$
$n(3) = 9$
$S^3 = \{s_1^3, \dots, s_9^3\}$</th><th>$t = 4$
$n(4) = 17$
$S^4 = \{s_1^4, \dots, s_{17}^4\}$</th></tr>
<tr><td rowspan="8">Terrible</td><td rowspan="4">Terrible</td><td rowspan="2">Terrible</td><td>Terrible</td></tr>
<tr><td>Extremely Poor</td></tr>
<tr><td rowspan="2">Very Poor</td><td>Very Poor</td></tr>
<tr><td>Worse than Poor</td></tr>
<tr><td rowspan="4">Poor</td><td rowspan="2">Poor</td><td>Poor</td></tr>
<tr><td>Rather Poor</td></tr>
<tr><td rowspan="2">Slightly Poor</td><td>Slightly Poor</td></tr>
<tr><td>Worse than Fair</td></tr>
<tr><td>Fair</td><td>Fair</td><td>Fair</td><td>Fair</td></tr>
<tr><td rowspan="8">Perfect</td><td rowspan="4">Good</td><td rowspan="2">Slightly Good</td><td>Better than Fair</td></tr>
<tr><td>Slightly Good</td></tr>
<tr><td rowspan="2">Good</td><td>Rather Good</td></tr>
<tr><td>Good</td></tr>
<tr><td rowspan="4">Perfect</td><td rowspan="2">Very Good</td><td>Better than Good</td></tr>
<tr><td>Very Good</td></tr>
<tr><td rowspan="2">Perfect</td><td>Excellent</td></tr>
<tr><td>Perfect</td></tr>
</table>

The semantic rule applied to assign a meaning to every label will be determined by a triangular linear function assigning a 3-tuple (a, b, c) to each label, where b represents the center of the triangle with a maximum membership value (i.e. 1), while a and c are the left and right ends of the triangular function defining the domain of the label concerned (Cabrerizo, Herrera-Viedma & Pedrycz, 2013; Pedrycz, 1994).

According to Zimmermann (2010), one way of presenting a fuzzy number is by using a parametric representation of its membership functions. A fuzzy set A in a universe of discourse U is defined as a set of pairs, as expressed in Equation (2):

$$A = \{(x, \mu_A(x)); x \in U\} \tag{2}$$

Here, $\mu_A : U \longrightarrow [0,1]$ is a membership function of the fuzzy set A. Thus, $\mu_A(x)$ –often written as $A(x)$– points out the degree of membership of the value $x \in U$ in the fuzzy set A. A membership function links elements x of a discourse domain U with elements of the interval $[0,1]$, which means that the closer $A(x)$ is to value 1, the greater the membership of object x in set A, whose terms are linearly and uniformly distributed with the triangular membership function shown in Equation (3):

$$\mu_A(x) = \begin{cases} 0 & ; \quad x < a \\ \dfrac{x-a}{b-a} & ; \; a \leq x \leq b \\ \dfrac{c-x}{c-b} & ; \; b \leq x \leq c \\ 0 & ; \quad x > c \end{cases} \tag{3}$$

Figure 1 illustrates the graphic representation of Table 1 using the triangular function previously described for the interval [0,1].

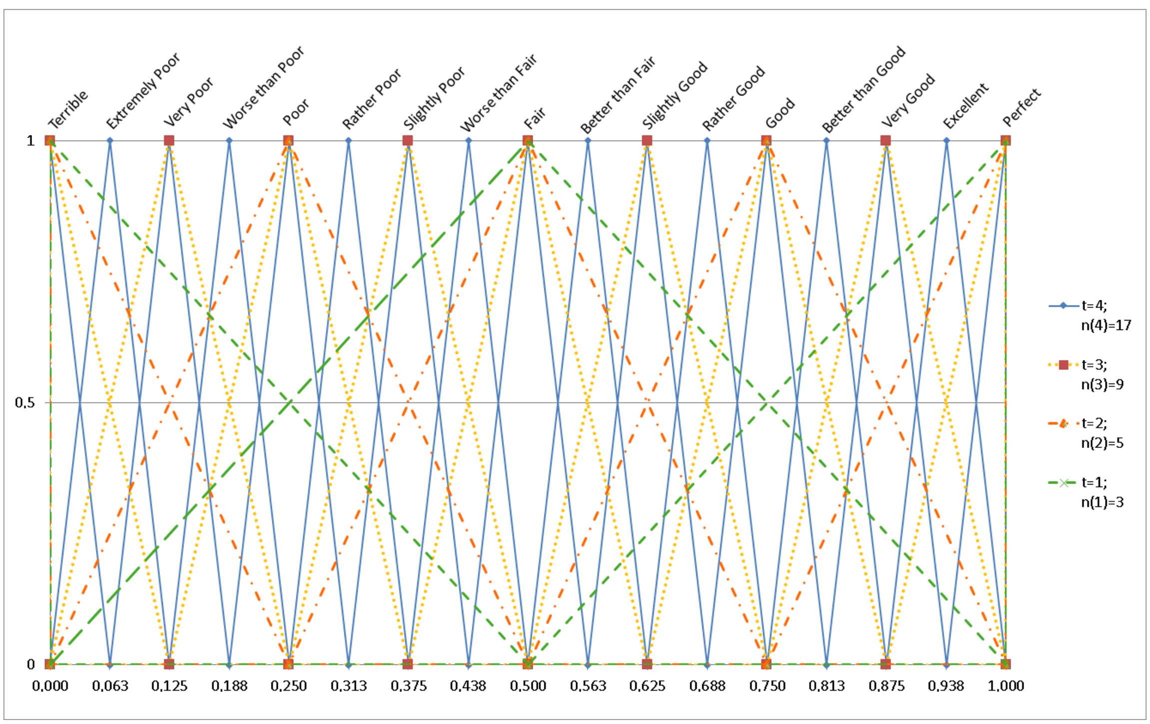


Fig. 1. Linguistic label sets for different t values.

However, the fact of using linguistic variables (based on natural language) involves a certain degree of uncertainty in the process, as we are dealing with words. This intrinsic difficulty of working with words would, in principle, involve a certain information loss; hence the need to find a form of representation that allows using these variables in the corresponding calculations, while it ensures that there will be no information loss. The model below allows working with linguistic labels and guarantees that no information will be lost in the process.

2.2 Linguistic 2-tuple representation model

This representation model was developed as a solution to the problem of the information loss in computational processes using words (Herrera & Martínez, 2000b), and it is based on the concept of symbolic translation explained below:

Definition 2: According to Herrera and Martínez (2000b), a linguistic representation using a 2-tuple can be defined as follows: Let $S = \{s_1, \dots, s_g\}$ be a linguistic term set and $\beta \in [1, g]$ a value representing the result of a symbolic aggregation operation (see Section 2.4 for more details about this), then the 2-tuple expressing the equivalent information to β is obtained with the function presented in Equation (4):

$$\Delta: [1, g] \to S \times [-0.5, 0.5)$$

$$\Delta(\beta) = (s_i, \alpha), \qquad \text{with} \begin{cases} s_i, & i = round(\beta) \\ \alpha = \beta - i, & \alpha \in [-0.5, 0.5) \end{cases} \tag{4}$$

where $round(\cdot)$ is the usual round operation, s_i is the label with the closest index to β, and α is the value of the symbolic translation.

Definition 3: Let $S = \{s_1, \dots, s_g\}$ be a linguistic term set and (s_i, α_i) a 2-tuple. There is always a Δ^{-1} function, shown in Equation (5), which returns its equivalent numerical value $\beta \in [1, g] \subset \Re$ from a 2-tuple:

$$\begin{gathered} \Delta^{-1}: S \times [-0.5, 0.5) \to [1, g] \\ \Delta^{-1}((s_i, \alpha)) = i + \alpha = \beta \end{gathered} \tag{5}$$

Hence, the conversion of a linguistic term into a linguistic 2-tuple consists in adding a zero (0) value as symbolic translation, as indicated in Equation (6):

$$s_i \in S \Rightarrow (s_i, 0) \tag{6}$$

The following section explains the 'linguistic hierarchy' concept and how the linguistic levels it contains are built. These principles are crucial for the operation of the proposed VER module.

2.3 Linguistic hierarchy

Some papers –like the ones published by Herrera and Martínez (2001), Martínez, Espinilla and Pérez (2008) and Wang (2008)– address the problem of handling linguistic variables with different granularity levels (that is, with a different number of labels). These papers establish a set of levels in which each level is made up of a set of linguistic terms with different granularity as compared to the other levels. Thus, each level in the linguistic hierarchy can be expressed as $l(t, n(t))$, where t is the level number and $n(t)$ is the granularity of the set of linguistic terms in the t level, i.e. $n(t)$ indicates the number of label of such set.

The levels within a hierarchy are ordered according to their granularity, so that successive levels can be represented as t and $t + 1$, provided that $n(t) < n(t + 1)$. Therefore, with this representation each level contains greater expressive richness as compared to the previous level.

A linguistic hierarchy LH can be defined as the union of all t levels, as expressed in Equation (7):

$$LH = \bigcup_t l\big(t, n(t)\big) \tag{7}$$

where the t level label set is represented as S^t. The following conditions need to be met to build a linguistic hierarchy:

1. Keep all the modal points of the membership function (the points where the function reaches its maximum membership value, i.e. 1) corresponding to each linguistic term, from the previous level to the next level in the hierarchy.

2. The transition between consecutive levels should result in a set of the kind S^{t+1}, adding a new term between every two terms of the t level set. This is done by reducing the size of each label's base (established with a triangular function), in order to ensure enough space for the new labels, which will be placed right in the middle of each pair of labels of the previous t level.

Therefore, the granularity of a $t+1$ level set of terms is obtained from its predecessor t level by Equation (8):

$$l(t, n(t)) \longrightarrow l(t+1, 2 \cdot n(t) - 1) \tag{8}$$

Figure 2 graphically represents an example of a four-level linguistic hierarchy (i.e. $t \in \{1, 2, 3, 4\}$) with a different granularity or number of labels $n(t)$ at each of its levels, i.e. $n(t) \in \{3, 5, 9, 17\}$.

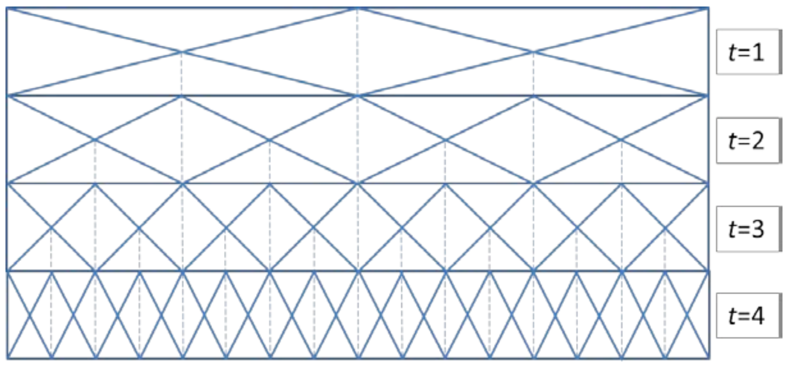


Fig. 2. Four-level linguistic hierarchy.

Linguistic hierarchies allow us to operate with labels from different levels without losing information, by using the 2-tuple representation model to transform the labels between hierarchy levels.

Definition 4: Let $LH = \bigcup_t l(t, n(t))$ be a linguistic hierarchy with the following term sets: $S^t = \{s_1^t, \dots, s_{n(t)}^t\}$. Equation (9) defines the function that transforms a term from the t level into a term belonging to the t' level using the 2-tuple linguistic representation (Herrera & Martínez, 2001):

$$TF_{t'}^t : l(t, n(t)) \longrightarrow l(t', n(t'))$$

$$TF_{t'}^t\big((s_i^t, \alpha^t)\big) = \Delta\left(\frac{\Delta^{-1}\big((s_i^t, \alpha^t)\big) \cdot (n(t') - 1)}{n(t) - 1}\right) \tag{9}$$

The term transformation function between different hierarchy levels is a bijective function, as indicated in Equation (10):

$$TF_t^{t'}\left(TF_{t'}^t\big((s_i^t, \alpha^t)\big)\right) = (s_i^t, \alpha^t) \tag{10}$$

This guarantees transformation without information loss.

Remark: We should point out that we are considering the decision-making framework where the universe of discourse U has no physical meaning, which allows us to use a different number of linguistic variables. If U had a physical sense, then linguistic terms used by experts would have typically strong practical connotations and any change (both as to the number and the meaning) would have a great risk of misunderstanding. In such a case, we could not use the concept of linguistic hierarchy to manage the different linguistic variables and we would have to apply other linguistic tools to manage such linguistic multi-granular contexts, such as the linguistic modelling based on fuzzy discrete numbers (Massanet et al. 2014).

2.4 Aggregation operator employed

When we have several opinions or evaluations from different people, it is very common in decision-making problems to aggregate those values within a unique value that will express the entire collective's opinion. As the arithmetic mean is a classical aggregation operator, its equivalent operator for linguistic 2-tuples is defined as follows:

Definition 5: Let $x = \{(r_1, \alpha_1), \ldots, (r_n, \alpha_n)\}$ be a set of 2-tuples; the arithmetic mean $\bar{x}$ of the elements of such set is computed by Equation (11):

$$\bar{x}((r_1, \alpha_1), \ldots, (r_n, \alpha_n)) = \Delta\left(\sum_{i=1}^{n} \frac{1}{n} \Delta^{-1}(r_i, \alpha_i)\right) = \Delta\left(\frac{1}{n}\sum_{i=1}^{n} \beta_i\right) \tag{11}$$

In this way, the arithmetic mean of a 2-tuple allows us to compute the mean of a set of linguistic values without any loss of information. One can find different types of aggregation operators in literature, depending on the needs of each case (Yager, 2007). In this case, we use the arithmetic mean aggregation operator with the 2-tuple representation model defined by Zhang (2012).

The following section explains the distance measurement applied to determine the most appropriate linguistic level to be used within the hierarchy, and hence the set of linguistic labels that will be used for expressing the final results.

2.5 Distance measurement

There are several measurement methodologies (Euclidean, Manhattan, t-norms, cosine function, etc.) for establishing the distance (difference) between two evaluations (Chiclana, García, Moral & Herrera-Viedma, 2013). In this case, and according to the principles of the base model development selected, we have opted for the Euclidean distance (Boran, Genç, Kurt & Akay, 2009; Su, Zeng & Ye, 2013), expressed in Equation (12) with the 2-tuple representation:

$$d_i = \Delta\left(\sum_{j=1}^{m}\left(\Delta^{-1}\big((s_{ij}, \alpha_{ij})\big) - \Delta^{-1}\big((s_{cj}, \alpha_{cj})\big)\right)^2\right)^{\frac{1}{2}} \forall i \in \{1, \ldots, n\} \tag{12}$$

where (s_{ij}, α_{ij}) values represent the assessments of the m criteria for the alternative A_i expressed as 2-tuples and (s_{cj}, α_{cj}) is the 2-tuple value chosen to calculate the distance to it for the c_j criterion.

Once we have all the necessary concepts and tools (linguistic variables, multi-granularity, 2-tuple representation model, aggregation operators and distance measurement), we can apply them to the model that will be used as a basis for the linguistic multi-criteria decision-making.

2.6 Base model employed: TOPSIS

The TOPSIS model is among the most widely used in DM processes (Jahanshahloo, Lotfi & Davoodi, 2009; Lai, Liu & Hwang, 1994; Shih, Shyur & Lee, 2007; Triantaphyllou, 2013), which is why it was selected as the basis model for the proposal at hand. This method, applied here as conceived initially, suggests a solution to DM problems by establishing a ranking of the different alternatives available through an analysis of the distance between each possible solution and the ideal and anti-ideal solutions. This approach can be expressed as follows: Let A_i, with $i \in \{1,2, \ldots, n\}$, be a set of solutions to a problem, where the set of evaluation criteria c_j, with $j \in \{1,2, \ldots m\}$, are taken into account, to which the evaluation weights w_j are applied. This model suggests that it is possible to build a decision matrix $x_{ij} = U_j(A_i)$, where U is the decision-maker's *usefulness* function that assesses the alternatives A_i based on the criteria c_j to maximize gains and minimize costs. The viability ranking of the different alternatives is determined through the interpretation of the results obtained in the calculation of proximity: the highest the proximity value obtained for a solution, the more desirable the solution to the DM problem.

The model is applied and explained in detail in the next section, using linguistic variables expressed with the 2-tuple representation model to avoid loss of information. Our proposal is set forth below, using all the concepts and foundations already presented and implementing them in this widely known and accepted base model.

3. Proposed model: LTOPSIS-2T-VER

In order to carry out this proposal, the TOPSIS model was used as a basis, with linguistic labels (L) and the 2-tuple representation model (2T), resulting in the LTOPSIS-2T model. Our idea may be implemented using nearly any MCDM model as a basis –either an existing or a specifically designed one– since our proposal aims at providing decision-makers with more understandable results obtained with any of the models mentioned, by applying the proposed VER module to the output generated by any of these models.

Figure 3 shows a basic diagram of the proposed model, taking the TOPSIS model as a basis and using linguistic labels represented as 2-tuples. It illustrates the process steps that allow establishing the best solution among the alternatives available for each case. The next section explains each process step in detail.

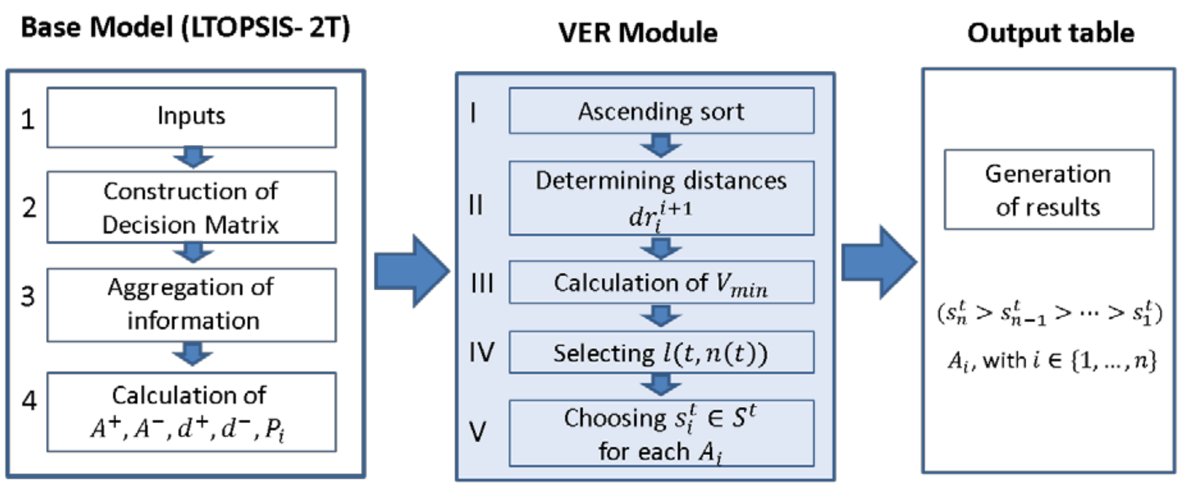


Fig. 3. Diagram of steps to be applied in the proposed LTOPSIS-2T-VER model.

3.1 Detailed explanation of the proposed process

It is worth noting that during the process we use linguistic labels converted to the 2-tuple representation model, to ensure that no information is lost. Once obtained the final proximity results, calculated with the base model employed, the VER module is applied (as shown in Figure 3) for establishing the level of membership within the linguistic hierarchy employed, and therefore the most appropriate label set. The process is completed with the conversion of results to the new set of linguistic labels, ranked from the most to the least significant.

3.1.1 Steps to be applied in the base model (LTOPSIS-2T)

This section presents the necessary steps for calculating the variables that will allow establishing, at the end of this procedure, the proximity value for each of these alternatives to the ideal solution. Since all the evaluations are expressed with linguistic labels (instead of using different evaluation scales), the calculation process of the TOPSIS model becomes significantly easier for not having to apply normalization procedures between different scales. As illustrated in Figure 3, these steps are:

1. Identify the model input information to be provided.
   a) Identify the set of possible alternative solutions $A = \{A_1, \ldots A_n\}$ in order to achieve the proposed goal (input 1).
   b) Establish the evaluation criteria $C = \{c_1, \ldots, c_m\}$ to be used for assessing the alternatives (input 2).

c) Estimate the importance (weight) of each evaluation criterion, $w = \{w_1, \dots, w_m\}$, taking into account that it is common for criteria to have different weights (input 3).

2. Build a decision matrix (criteria/alternatives) for each expert in the set $E = \{e_1, \dots, e_p\}$. Each element of these matrixes will be a label or linguistic term from one of the hierarchy subsets represented by a triangular fuzzy number. It is advisable to use the granularity corresponding to level $t = 2$ in order to make experts' work easier, which will result in a subset $S^2 = \{s_1^2, \dots, s_5^2\}$ made up of five linguistic labels. Table 2 shows the structure of each of these decision matrixes.

Table 2. Structure of the Criteria/Alternatives decision matrixes.

Weights	$\boldsymbol{w_1}$	$\boldsymbol{w_2}$	$\cdots$	$\boldsymbol{w_m}$
Criteria / Alternatives	$\boldsymbol{c_1}$	$\boldsymbol{c_2}$	$\cdots$	$\boldsymbol{c_m}$
$\boldsymbol{A_1}$	x_{11}	x_{12}	$\cdots$	x_{1m}
$\boldsymbol{A_2}$	x_{21}	x_{22}	$\cdots$	x_{2m}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\boldsymbol{A_n}$	x_{n1}	x_{n2}	$\cdots$	x_{nm}

3. Aggregate the information contained in the matrixes relating to experts e_k, $\forall k \in \{1, \dots, p\}$, to obtain a unified matrix of expert opinions. The evaluations x_{ij}, $\forall i \in \{1, \dots, n\}$, $\forall j \in \{1, \dots, m\}$, contained in the resulting matrix are expressed by means of 2-tuples (s_{ij}^t, α^t). In this proposal, all the experts are considered to have the same level of knowledge (level of importance). The linguistic arithmetic mean aggregation operator shown in Equation (13) is used at this step:

$$\bar{x}^e((s_{11}^t, \alpha_{11}^t)_1, \dots, (s_{nm}^t, \alpha_{nm}^t)_k) = \Delta\left(\frac{1}{p}\sum_{k=1}^{p} \Delta^{-1}\left(\left(s_{ij}^t, \alpha_{ij}^t\right)_k\right)\right), \forall i \in \{1, \dots, n\}, \forall j \in \{1, \dots, m\} \quad (13)$$

The matrix obtained is multiplied by the weights corresponding to each criterion, getting a weighted matrix with the structure presented in Equation (14):

$$\bar{X} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & \cdots & c_m \end{array} \\ \begin{bmatrix} \bar{x}_{11} & \bar{x}_{12} & \dots & \bar{x}_{1m} \\ \bar{x}_{21} & \bar{x}_{22} & \dots & \bar{x}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{x}_{n1} & \bar{x}_{n2} & \dots & \bar{x}_{nm} \end{bmatrix} \\ \begin{array}{cccc} w_1 & w_2 & \cdots & w_m \end{array} \end{array} \quad (14)$$

where $\bar{x}_{ij}$, $\forall i \in \{1, \dots, n\}, \forall j \in \{1, \dots, m\}$, is the aggregated element corresponding to alternative A_i, with the criteria c_j and weight w_j.

4. Calculate the parameters used by the base model.
   a) Establish the positive ideal solution (A^+) and the negative ideal solution (A^-) from the unified matrix obtained in the previous step (3). Equation (15) and Equation (16) will be used in this operation:

$$A^+ = \{(\max_i(\bar{x}_{ij}) \mid j \in Y), (min_i(\bar{x}_{ij}) \mid j \in Z)\} = \{\bar{x}_1^+, \bar{x}_2^+, \dots, \bar{x}_n^+\} \quad (15)$$

$$A^- = \{(\min_i(\bar{x}_{ij}) \mid j \in Y), (\max_i(\bar{x}_{ij}) \mid j \in Z)\} = \{\bar{x}_1^-, \bar{x}_2^-, \dots, \bar{x}_n^-\} \quad (16)$$

$$\forall i \in \{1, \dots, n\}, \forall j \in \{1, \dots, m\}$$

   where Y is associated with gain criteria (maximum values) and Z is associated to cost criteria (minimum values).

   b) Obtain the distances (d^+, d^-) to the ideal solutions (A^+ and A^-) for each alternative, obtained in the previous step (4a). The weights (levels of importance) of each criterion established in step 1c have been applied in step 3, when calculating the weighted matrix. In order to obtain the distance values, Equation (17) and Equation (18) –using the conversions defined for the 2-tuple representation– are applied:

$$d_i^+ = \Delta\left(\sum_{j=1}^{m}\left(\left|\Delta^{-1}\left((\bar{x}_{ij},\alpha_{ij})\right) - \Delta^{-1}\left((\bar{x}_j^+,\alpha_j^+)\right)\right|\right)^2\right)^{0.5}, \qquad \forall i \in \{1,\dots,n\} \tag{17}$$

$$d_i^- = \Delta\left(\sum_{j=1}^{m}\left(\left|\Delta^{-1}\left((\bar{x}_{ij},\alpha_{ij})\right) - \Delta^{-1}\left((\bar{x}_j^-,\alpha_j^-)\right)\right|\right)^2\right)^{0.5}, \qquad \forall i \in \{1,\dots,n\} \tag{18}$$

c) Calculate the proximity coefficient (P_i) for each alternative, represented with a 2-tuple. This involves establishing the position for each alternative, taking into account the distance (d^+, d^-) to the best and worst solution (A^+, A^-). Equation (19) is used for this calculation:

$$P_i = \Delta\left(\frac{\Delta^{-1}(d_i^-)}{\Delta^{-1}(d_i^+) + \Delta^{-1}(d_i^-)}\right), \qquad \forall i \in \{1,\dots,n\} \tag{19}$$

3.1.2 Steps to apply in the VER module

This module allows applying a variable expressive richness (VER) to the results obtained by the base model, which will be reflected in the proposed model output. The steps to take in this module are the following ones:

I. Rank the proximity results, $r_i = \Delta^{-1}(P_i), \forall i \in \{1,\dots,n\}$, corresponding to each alternative (obtained in the step 4c of the previous subsection) in ascending order, i.e. $r_1 < r_2 < r_3 < \cdots < r_n$.

II. Calculate the distance (dr_i^{i+1}) between the consecutive pairs of results, as shown in Equation (20).

$$dr_i^{i+1} = (|r_i - r_{i+1}|^2)^{\frac{1}{2}}, \qquad \forall i \in \{1,\dots,n-1\} \tag{20}$$

where dr_i^{i+1} is the absolute difference between the initial value (r_i) and the following value (r_{i+1}) of the results previously ranked in ascending order.

III. Determine the minimum value of the $n-1$ results obtained in the previous step (II), by applying Equation (21).

$$V_{min} = \min\left(dr_i^{i+1}\right), \qquad \forall i \in \{1,\dots,n-1\} \tag{21}$$

IV. Determine the most appropriate set of linguistic labels among the available sets in the LH linguistic hierarchy, to represent linguistically the proximity results obtained, by applying the following rule:

$$\text{If } V_{min} \le s_1^t(c) \text{ and } t = q, \text{ then } V_{min} = s_1^q(c),$$

$$\text{else if } s_1^t(c) < V_{min} \le s_1^{t-1}(c), \text{ then } V_{min} = s_1^{t-1}(c), \text{ with } t \in \{1,\dots,q-1\},$$

where c represents the right end of the triangular function defining the corresponding s_1^t label domain.

The V_{min} value will be compared with the values at the base of the first s_1^t label on each level ($\forall t \in \{1,\dots,n\}$) of the linguistic hierarchy (see Figure 4, where the interval considered is [0,0.5]). This comparison will allow determining the interval that contains the calculated V_{min} value, and hence the t level that best represents the results obtained. The rational for using this value to determine the level of the linguistic hierarchy (and consequently, the set of labels) to use to provide the final result is that such value determines how distant the closest results provided by the base model (TOPSIS, in the case presented in this paper) are. If this value is very small, it means that a set with a greater number of linguistic labels should be used so that each alternative is labelled with a different label, whereas if that value is greater, it will suffice to use a set with fewer labels to guarantee that goal.

V. Apply the S^t set of labels selected in the previous step (IV) to the results generated by the base model, taking into account the closest label to each result obtained. This can be expressed by means of Equation (22):

$$s_i^t(b) = round(r_i), \qquad \forall i \in \{1, \dots, n(t)\}, \forall t \in \{1, \dots, q\} \tag{22}$$

where $round$ is the standard rounding function that outputs the b value representing the central point of the triangular function for the s_i^t label closest to r_i, so that t designs the level (1, 2, 3 or 4, in our example) of the label set within the hierarchy, while $n(t)$ is the maximum number of linguistic labels (3, 5, 9 or 17, in our example) corresponding to that level. In this way, it is possible to represent the results using more appropriate descriptive labels for the model output, as well as more representative and understandable by decision-makers.

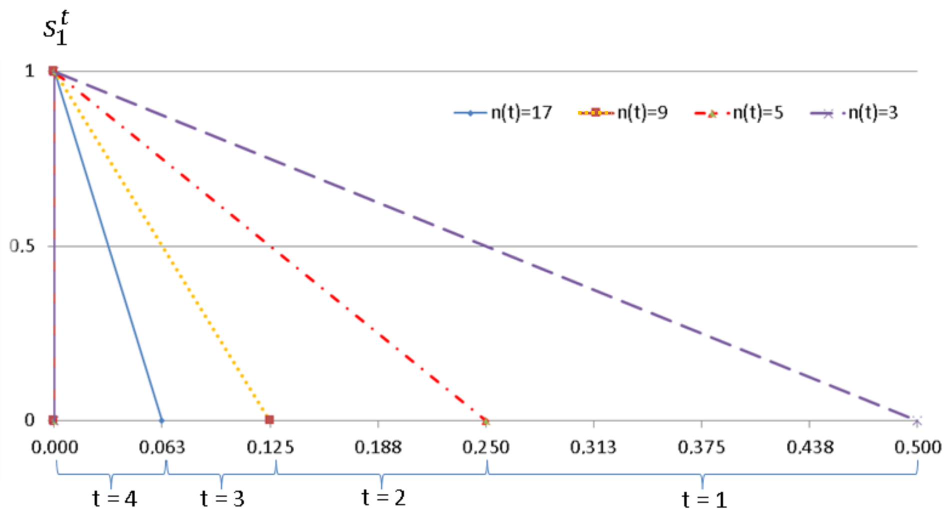


Fig. 4. Diagram of the s_1^t label for each of the $t \in \{1,2,3,4\}$ levels employed for determining the ideal expressive richness for each case in the example put forward.

3.1.3 Generation of final output results

The output table will be made up by all the alternatives in the $A = \{A_1, \dots A_n\}$ set, assessed with the corresponding linguistic label of the selected linguistic level ($s_i^t \in S^t$), according to the required or the most appropriated expressive richness.

The results are ranked in descending order according to the values of their labels ($s_n^t > s_{n-1}^t > \cdots > s_1^t$), the highest label being associated to alternative A_i, with $i \in \{1, \dots, n\}$, which obtained a higher value in the proximity calculation (P_i).

3.2 Expression of results as compared with other models

Table 3 displays a comparison of different characteristics (more concretely, 8) concerning the expression of results in the main models most used in MCDM problems, and especially in qualitative ones, with the aim of understanding the differences between them in this respect (expression of output results). The last column corresponds to the new model put forth in this paper.

As common models provide numerical results, the decision maker requires additional information (such as scale used, maximum value, minimum value, context, etc.) for an adequate interpretation of those results. Although to a lesser extent, the 2-tuples linguistic model also requires some interpretation, due to its numerical component α, which represents the symbolic translation value with respect to the linguistic label (first component of the 2-tuple). However, the results expressed through linguistic labels do not require interpretation by the decision maker, because they are in line with his/her way of thinking.

Table 3. Comparison of characteristics related to the expression of results in different models.

Features / Models	Common	Linguistic	Linguistic 2-tuple	Linguistic VER
Expression of results	Numerical	Linguistic labels	Linguistic labels in 2-tuples	With variable expressive richness
Diversity in the results	Positive real numbers	Subset of labels	Subset of labels	Group of label subsets
It requires interpretation of results	Yes	No	Yes	No
Applicability to qualitative problems	Limited	Good	Good	Very good
Use of natural language	No	Limited by the number of labels used	Limited by the number of labels used	Limited by the group of label subsets used
Number of linguistic subsets	-	1	1	Several
Linguistic auto-setting	No	No	No	Yes
Granularity depending on the outcome	-	No	No	Yes

With respect to qualitative problems, numerical (quantitative) models have a more limited applicability, while linguistic ones have a better applicability to this type of problems, being our proposal the most adequate, due to it provides a greater richness and flexibility when expressing the results, which makes them more differentiated and, consequently, understandable for the decision maker.

As shown in Table 3, there are several factors that provide a significant advantage in the presentation of results with the VER module over the other models included in such table. Thus, for example, it can be noted that our model is the only one that presents the novel features of “Linguistic auto-setting” and “Granularity depending on the outcome”, which allow that the linguistic labels used for expressing results are dynamically adapted to the context of the case concerned, as well as differentiating nearby results. This is entirely in line with humans’ inherent capacity to communicate their preferences using natural language and with their ability to choose the most appropriate adjectives in every case.

Below, a case study putting into practice what has been explained so far.

4. Application examples of the proposed model

The first part of this section is dedicated to the detailed application of our model in a case study related to the Services sector, in particular to the Information and Communication Technologies (ICT) sector. In the second part, the results obtained in the case study are analyzed and, finally, in the third part, the results obtained by applying our model to other three case studies are compared with those obtained with other linguistic MCDM models.

4.1 Detailed application of the LTOPSIS-2T-VER model to a case study

A company in the ICT sector is facing a decision-making problem: choosing the products that are a matter of priority in terms of investment for the next six-months. A group of experts $E = \{e_1, \dots, e_3\}$ has been selected for expressing their preferences in this regard. All experts are assumed to have the same level of expertise, so their opinions will have the same level of importance.

The following alternatives are available, expressed by the set $A = \{A_1, \dots, A_5\}$:

- A_1: Purchase a new range of smart terminals (smartphones).
- A_2: Acquire new satellite capacity to increase TX[2] redundancy.
- A_3: Extend the free Internet network (Wi-Fi) to shopping centers, stadiums and public places in provincial capitals.
- A_4: Invest in infrastructure for new customer service offices.
- A_5: New prime time advertising campaign to promote new n-P (n-Play)[3] services.

[2] Data transmission systems used by telecommunication operators.

The following set of criteria $C = \{c_1, \dots, c_4\}$ has to be analyzed by experts for each alternative:

- c_1: Financial risk
- c_2: Expandability
- c_3: Social and political impact
- c_4: Environmental impact

Depending on their importance, each of these criteria will be assigned a weight determined by the set $W = \{w_1, \dots, w_4\}$.

The proposed model was applied to this case study, taking the LTOPSIS model as a basis and adding the VER module to express output results in the most appropriate way. All the data gathered under this case study were expressed with linguistic labels and entirely processed through the 2-tuple linguistic representation, in order to ensure uniformity throughout the process.

Table 4 shows the evaluations expressed in natural language by each of the (3) participating experts, who assessed all the alternatives from the perspective of each criterion. The labels used were from level $t = 2$, belonging to the $S^2 = \{s_1^2, \dots, s_5^2\}$ set, where $s_1^2 = Strongly\ Disagree$ (SD), $s_2^2 = Disagree$ (D), $s_3^2 = Neutral$ (N), $s_4^2 = Agree$ (A), and $s_5^2 = Strongly\ Agree$ (SA). The weights assigned to each criterion were expressed with a set containing the same number of terms, with the following labels: $s_1^w = Not\ Important$ (NI), $s_2^w = Little\ Importance$ (LI), $s_3^w = Neutral$ (N), $s_4^w = Important$ (I), and $s_5^w = Very\ Important$ (VI) (see Table 5).

Table 4. Assessment matrix of each expert (e_1, e_2 and e_3) for this case study.

e_1	c_1	c_2	c_3	c_4
A_1	A	SA	N	N
A_2	N	D	A	N
A_3	SA	A	SA	SA
A_4	D	N	N	SA
A_5	SA	SA	D	A

e_2	c_1	c_2	c_3	c_4
A_1	N	A	D	SD
A_2	SD	SA	SA	SA
A_3	A	SA	N	SA
A_4	SA	SD	SD	A
A_5	N	SD	A	SD

e_3	c_1	c_2	c_3	c_4
A_1	SA	N	SD	D
A_2	SD	SA	N	SA
A_3	D	SD	SA	N
A_4	SA	D	SA	SD
A_5	A	SD	A	SA

Table 5. Weights assigned by the expert group to each criterion, expressed with linguistic labels.

	c_1	c_2	c_3	c_4
w_j	LI	I	VI	NI

Table 6 shows the matrix resulting from the aggregation of the three experts' opinions, for which we applied the equation developed in Definition 5. That information is expressed in natural language, using the 2-tuple representation.

Table 6. Matrix of the aggregation of the three experts' opinions, expressed with 2-tuples.

A_i / c_j	c_1	c_2	c_3	c_4
A_1	(A,+0.000000)	(A,+0.000000)	(D,+0.000000)	(D,+0.000000)
A_2	(D,-0.083340)	(A,+0.000000)	(A,+0.000000)	(A,+0.083300)
A_3	(A,-0.083400)	(N,+0.083300)	(A,+0.083300)	(A,+0.083300)
A_4	(A,+0.000000)	(D,+0.000000)	(N,+0.000000)	(N,+0.083300)
A_5	(A,+0.000000)	(D,+0.083300)	(N,+0.083300)	(N,+0.083300)

Figure 5 shows a diagram of the proposed model, implemented using IBM's SPSS Modeler[4] tool, where the VER module –essential in this proposal– is highlighted in a box. Note that this tool represents each subroutine or subprocess via a star symbol, so this is an abstraction mechanism to hide the process carried out in the corresponding module. Consequently, we will show the content of the VER module (which is the main contribution of our proposal) below, in additional figures (7, 8 and 9), which will be conveniently explained.

[3] Market package offered by telecommunication operators to their users, normally including voice services (landline and mobile), broadband and mobile Internet Access, television, VoD (video on demand), etc.

[4] http://www-03.ibm.com/software/products/en/spss-modeler

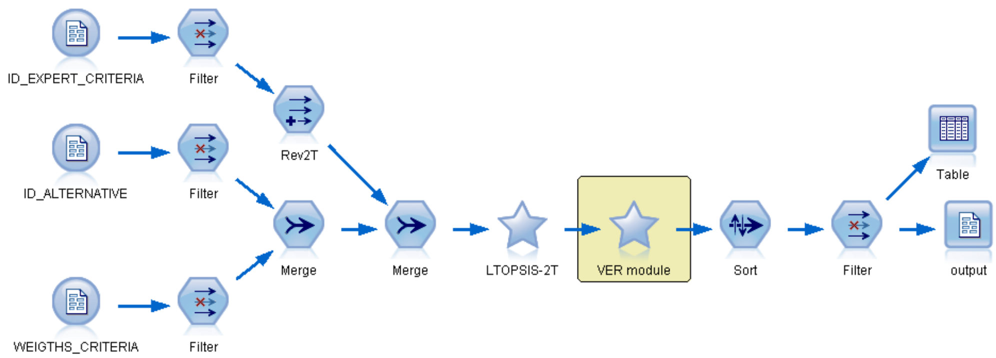
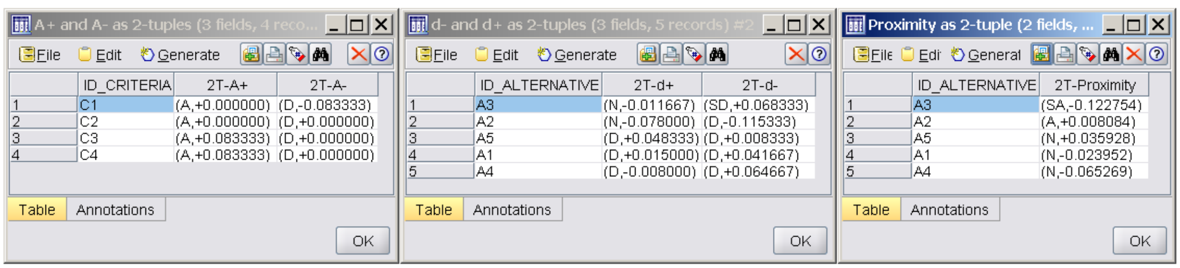


Fig. 5. Model developed using IBM's SPSS Modeler software, highlighting the VER module.

As shown in Figure 5, once we have the results generated by the base DM model used (in this case, LTOPSIS-2T), the process enters the VER module which analyses and differentiates the results previously obtained (by the base model) so that their final expression is as rich as possible.

Figure 6 shows the results obtained at the output of the base DM model used (LTOPSIS-2T, in our case) for the ideal positive and negative solutions (A^+, A^-), as well as the distance between each alternative and the ideal positive (d^+) and negative (d^-) solutions. The calculated proximity value (P) is also shown. As can be seen, all the resulting values are expressed using the linguistic 2-tuple representation.


A+ and A- as 2-tuples (3 fields, 4 reco...

	ID_CRITERIA	2T-A+	2T-A-
1	C1	(A,+0.000000)	(D,-0.083333)
2	C2	(A,+0.000000)	(D,+0.000000)
3	C3	(A,+0.083333)	(D,+0.000000)
4	C4	(A,+0.083333)	(D,+0.000000)

d- and d+ as 2-tuples (3 fields, 5 records) #2

	ID_ALTERNATIVE	2T-d+	2T-d-
1	A3	(N,-0.011667)	(SD,+0.068333)
2	A2	(N,-0.078000)	(D,-0.115333)
3	A5	(D,+0.048333)	(D,+0.008333)
4	A1	(D,+0.015000)	(D,+0.041667)
5	A4	(D,-0.008000)	(D,+0.064667)

Proximity as 2-tuple (2 fields, ...

	ID_ALTERNATIVE	2T-Proximity
1	A3	(SA,-0.122754)
2	A2	(A,+0.008084)
3	A5	(N,+0.035928)
4	A1	(N,-0.023952)
5	A4	(N,-0.065269)

Fig. 6. Results obtained for A^+, A^-, d^+, d^- and proximity P, expressed with linguistic 2-tuples (2T-A+, 2T-A-, 2T-d+, 2T-d- and 2T-Proximity, respectively).

The content of the VER module is shown in Figure 7, while Figures 8 and 9 display the implementation of the two sub-modules contained in such module.

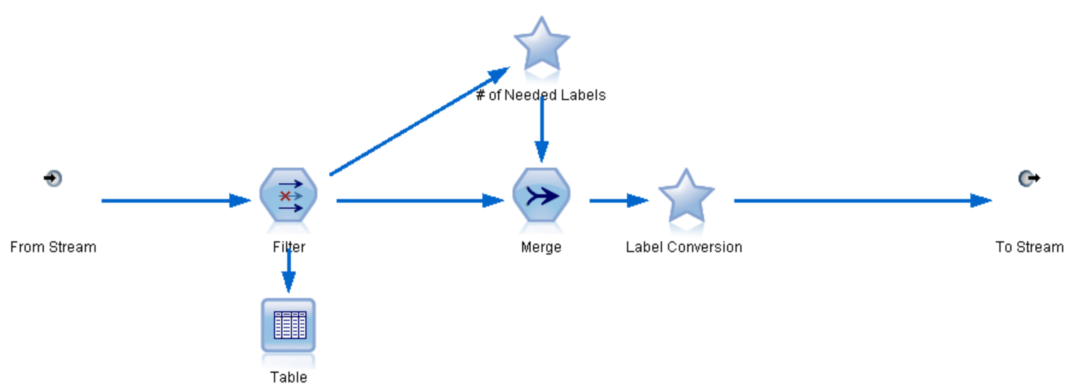


Fig. 7. Implementation details of the VER module.

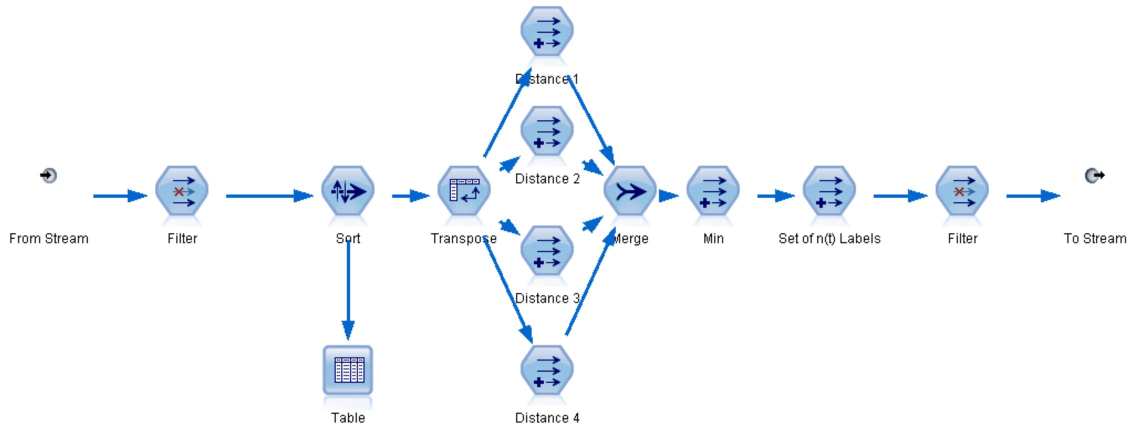


Fig. 8. Implementation of the *# of Needed Labels* sub-module. It determines the V_{min} value and the s_1^t label.

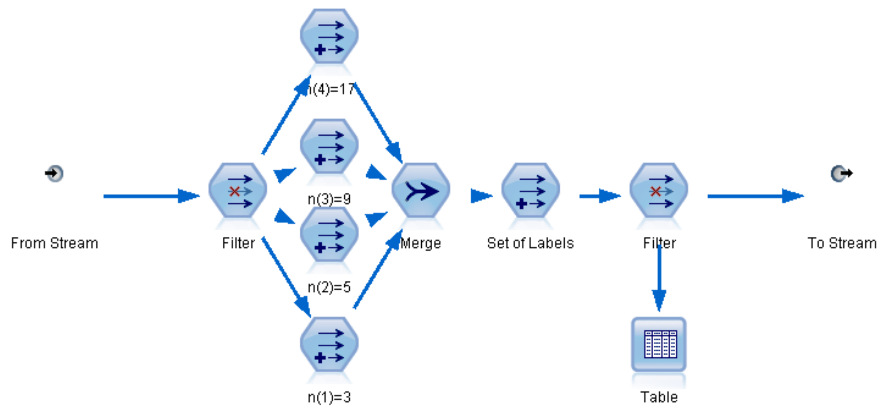


Fig. 9. Implementation of the *Label Conversion* sub-module. It determines the set of labels to be used.

To clarify the application of the steps to be executed in the VER module, which is the main contribution of our proposal, the main partial results obtained by applying each of these steps (see subsection 3.1.2 for more details) are presented below.

After applying step I to our case study, we get the results presented in Figure 10, just at the output of the *Sort* element (shown in Figure 8).

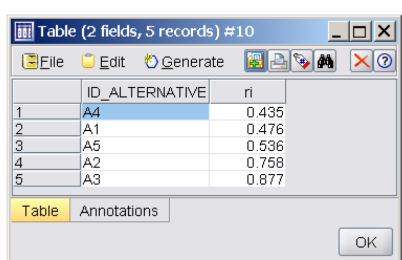


Table (2 fields, 5 records) #10

File Edit Generate

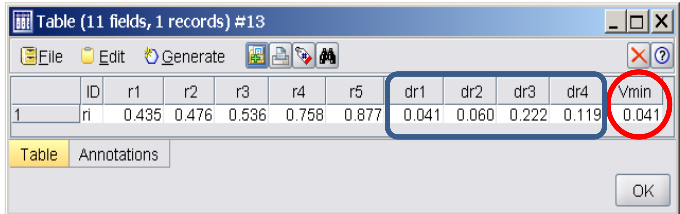
	ID_ALTERNATIVE	ri
1	A4	0.435
2	A1	0.476
3	A5	0.536
4	A2	0.758
5	A3	0.877

Table Annotations

OK

Fig. 10. Values of r_i in ascending order obtained after applying step I.

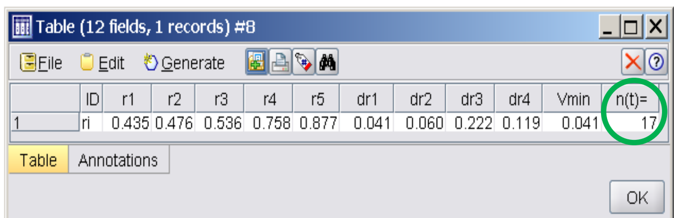
After applying steps II and III, we obtain the results shown in Figure 11, just at the output of the *Min* element (shown in Figure 8). Note that the values enclosed with a blue line rectangle correspond to the distances calculated in step II, while the value surrounded by a red line circle corresponds to the minimum value (V_{min}) of such distances. This latter value is used to determine the most appropriate level of the linguistic hierarchy to be used in this particular case study.



	ID	r1	r2	r3	r4	r5	dr1	dr2	dr3	dr4	Vmin
1	ri	0.435	0.476	0.536	0.758	0.877	0.041	0.060	0.222	0.119	0.041

Fig. 11. Values obtained for the distances dr_i^{i+1} between r_i used to determine V_{min}.

Figure 12 shows the results obtained after applying step IV, just at the output of the *Set of n(t) Labels* element (shown in Figure 8). Once determine the value V_{min}, this is compared to the different values of t (shown in the horizontal axis in Figure 4) to determine the most appropriate number of linguistic labels (surrounded with a green line circle in Figure 12) to represent the results calculated by the base model.

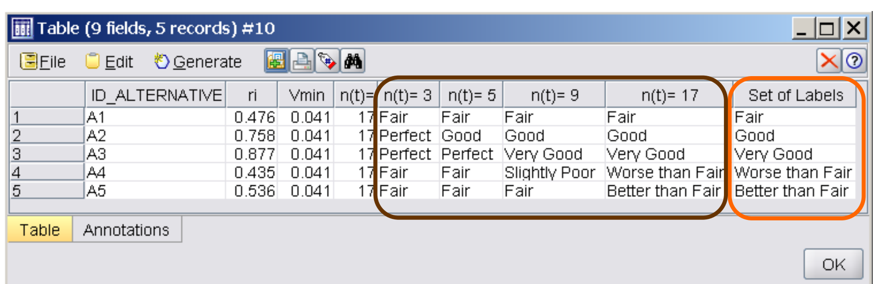


	ID	r1	r2	r3	r4	r5	dr1	dr2	dr3	dr4	Vmin	n(t)=
1	ri	0.435	0.476	0.536	0.758	0.877	0.041	0.060	0.222	0.119	0.041	17

Fig. 12. Determining the most appropriate number of labels $n(t)$ to be used for the V_{min} value previously calculated.

The subroutine shown in Figure 9 is in charge of applying step V, i.e. assigning the corresponding labels to each alternative considered, according to the $n(t)$ value previously determined. These labels are shown in Figure 13, which is obtained at the output of the *Filter* element shown in Figure 9.

Note that the proposed algorithm compares the labels assigned to each alternative in each of the levels that make up the linguistic hierarchy considered (see the columns enclosed by the brown line box in Figure 13) to determine the minimum of such levels in which all the assigned labels are different or where there is a greater differentiation in them. The set of labels corresponding to that level will be the one chosen to be provided as the result (last column, surrounded by the orange line box in Figure 13). Thus, there will be less ambiguity in the results provided and hence a greater ease to interpret them is offered to the decision maker.



	ID_ALTERNATIVE	ri	Vmin	n(t)=	n(t)= 3	n(t)= 5	n(t)= 9	n(t)= 17	Set of Labels
1	A1	0.476	0.041	17	Fair	Fair	Fair	Fair	Fair
2	A2	0.758	0.041	17	Perfect	Good	Good	Good	Good
3	A3	0.877	0.041	17	Perfect	Perfect	Very Good	Very Good	Very Good
4	A4	0.435	0.041	17	Fair	Fair	Slightly Poor	Worse than Fair	Worse than Fair
5	A5	0.536	0.041	17	Fair	Fair	Fair	Better than Fair	Better than Fair

Fig. 13. Label assignment to each alternative considered, according to the $n(t)$ value previously calculated.

The final result after process completion (i.e., the contents of the *output* element displays in Figure 5) is shown in Figure 14. Note that the order presented in this last Figure is according to the importance degree obtained by the corresponding alternative, i.e. from the best label to the worse one.

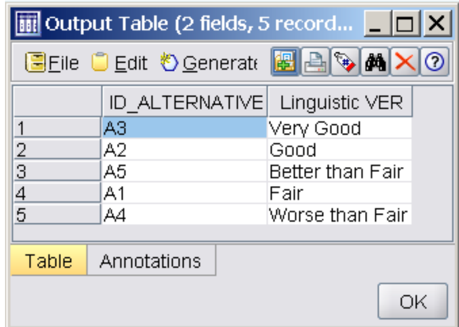


Fig. 14. Final results obtained with the VER module.

4.2 Analysis of results for the case study depicted

This section presents the results obtained and analyzes the advantage of the solution put forth over the other forms of expressing results. Table 7, which shows the results obtained for the case study explained in Section 4.1, contains the results linguistically expressed in three different ways (Linguistic, Linguistic 2-tuples and Linguistic VER).

Table 7. Results obtained for the case study depicted, expressed with different output types.

Results	Linguistic	Linguistic 2-tuples	Linguistic VER	Labels (s_i^t)	Observation
A_3	Strongly Agree	(SA,-0.122754)	Very Good	s_{15}^4	The VER module is automatically adjusted in order to provide the best response. This example uses $n(t) = 17$ linguistic labels
A_2	Agree	(A,+0.008084)	Good	s_{13}^4	
A_5	Neutral	(N,+0.035928)	Better than Fair	s_{10}^4	
A_1	Neutral	(N,-0.023952)	Fair	s_9^4	
A_4	Neutral	(N,-0.065269)	Worse than Fair	s_8^4	

As shown in Table 7, the results expressed in natural language using a set of 5 labels (second and third columns) can be confusing when it comes to selecting a final solution. This is due to the fact that the same linguistic label or evaluation might have been assigned to more than one alternative. However, looking at the results contained in the fourth column, we can see how answers are more varied thanks to the automatism implemented in the VER module and they provide clearer information to the decision maker, which allows his/her to choose without hesitation the most appropriate alternative. In other words, it allows making better and faster decisions.

The VER module could also pose the problem of having two different alternatives assessed with the same label, which would point out the need to add an extra level to the hierarchy, with new adjectives or linguistic expressions. It could also happen, depending on the nature of the problem concerned, that both alternatives are accepted as possible solutions.

4.3 Comparative analysis of results obtained in other case studies

This section presents the results of three other real cases (shown in Table 8) where the proposal put forward in this paper was applied, as well as two other linguistic models, and compares the results obtained in each one. These results are the outcome of DM problems considering five possible alternative solutions, where five linguistic labels are used for assessing every alternative. In the column assigned to our proposal (Linguistic VER) we can see that different $n(t)$ granularity was used for representing the final assessments. This diversity implies that the module will not necessarily use the same labels to represent results for different problems, since this is rather determined by the distance between the results obtained.

Table 8. Comparison of the results obtained by applying 3 different models to 3 real examples considering 5 alternatives. The results are therefore expressed with different output expressions.

	Real-life examples (5 alternatives each)	**Linguistic**	**Linguistic 2T**	**Linguistic VER**	**Labels (s_i^t)**	**Observation**
1	Analysis of a business plan	Agree Agree Agree Neutral Neutral	(A,-0.000200) (A,-0.039800) (A,-0.093800) (N,+0.034000) (N,+0.022300)	Good Rather Good Slightly Good Better than Fair Fair	s_{13}^4 s_{12}^4 s_{11}^4 s_{10}^4 s_9^4	The VER module is automatically adjusted to provide a better response. This example uses $n(t) = 17$ labels.
2	Purchase of a web server	Agree Agree Disagree Disagree Strongly Disagree	(A,-0.000269) (A,-0.063000) (D,+0.081000) (D,-0.097000) (SD,+0.061400)	Good Slightly Good Slightly Poor Very Poor Terrible	s_7^3 s_6^3 s_4^3 s_2^3 s_1^3	The VER module is automatically adjusted to provide a better response. This example uses $n(t) = 9$ labels.
3	Evaluation of cloud computing services	Strongly Agree Agree Neutral Neutral Disagree	(SA,-0.094373) (A,+0.081770) (N,-0.018757) (N,-0.056272) (D,+0.091149)	Very Good Better than Good Fair Worse than Fair Rather Poor	s_{15}^4 s_{14}^4 s_9^4 s_8^4 s_6^4	The VER module is automatically adjusted to provide a better response. This example uses $n(t) = 17$ labels.

Examples 1 and 2 (first two rows) displayed in Table 8 show how the first two models (Linguistic and Linguistic 2T) only apply 2 and 3 different labels in each example, respectively, from the 5 labels available for expressing results, so that several of the 5 alternatives analyzed in each example are assessed with the same label. On the contrary, the Linguistic VER model uses a different linguistic label for expressing the results obtained for each of the 5 alternatives. It is worth noting that by applying this latter model in the three examples shown in Table 8, we prevent label repetition in the final assessment of every solution.

5. Conclusions and future work

The linguistic MCDM model based on variable expressive richness (VER) presented in this paper introduces several advantages (detailed below) that make decision making easier. Besides, this proposal expresses results through different linguistic expressions or labels, so that the language employed is understood by any expert involved in the DM process, regardless of their area of knowledge or work in a company. In this way, we reduce the uncertainty inherent in DM problems, as well as the response times, thus substantially improving DM efficiency.

The main novelty introduced by this model is the self-detection of the most appropriate label set for expressing solutions (assessment of the different alternatives) to DM problems in every case in the most flexible, adequate and expressive way possible.

The proposed solution introduces the following advantages:

- Output results are totally expressed within the natural language framework, through the use of linguistic labels.
- Use of a multilevel linguistic hierarchy made up of label sets with different granularity.
- A smart system that assesses each alternative available based on the optimization model, which self-detects the most appropriate labels in each case.
- Indication of the label subset to be applied in every case for the best expression of results.
- The granularity used in the input by experts to assess the criteria and express the weights has no effects on the output granularity provided by the VER module.
- Independent of the input type, which can be numerical, linguistic, fuzzy numbers, etc.
- Compatible with the use of input multi-granularity.
- Applicable to multiple types of results (numerical, linguistic, 2-tuple, etc.) generated by different MCDM models.
- Modular and flexible model, adaptable to different DM requirements and problems.
- No need of applying normalization processes between the different (numerical) units used to express the various dimensions used in a given case study, since our proposal process all the multidimensional information expressed linguistically by means of linguistic labels.

As a future line of research, this development could be extrapolated to fuzzy models with different membership functions, as well as to models with multi-granular input. Moreover, we consider that the model proposed here can be incorporated into fuzzy multidimensional models, as proposed by Carrasco et al. (2013), in order to facilitate its practical use.

Acknowledgments

The authors would like to acknowledge the financial support received from the European Regional Development Fund (ERDF) for the Research Projects TIN2016-75850-R, TIN2016-79484-R and TIN2013-40658-P.

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