Stability of time series models based on weakening buffer operators

Journal:	*Journal of Systems Science & Complexity*
Manuscript ID	JSSC-2023-0142
Manuscript Type:	Original Article
Date Submitted by the Author:	12-Apr-2023
Complete List of Authors:	Li, Chong; Civil Aviation Flight University of China, School of Air Traffic Management Yang, Yingjie; De Montfort University, School of Computer Science and Informatics Zhu, Xinping; Civil Aviation Flight University of China, School of Air Traffic Management
Keywords:	Time series, Sequence operator, Model stability, Model perturbation analysis
Speciality:	Forecasting Science, Statistics and Data Science

J Syst Sci Complex (20XX) XX: 1–29

Stability of Time Series Models Based on Weakening Buffer Operators*

LI Chong · YANG Yingjie · ZHU Xinping

DOI:
Received: x x 2023 / Revised: x x 2023

Abstract Different weakening butter operators in time series model analysis usually result in different model sensitivity, which sometimes affects the effectiveness of relevant operator-based methods. In this paper, the stability of two classic weakening buffer operator-based series models is studied; then a new data preprocessing method based on a novel fractional bidirectional weakening buffer operator is provided, whose effect in improving model stability is tested and utilized in prediction problems. Practical examples are employed to demonstrate the efficiency of the proposed method in improving model stability in noise scenarios. The comparison indicates that the proposed method overcomes the disadvantage of many weakening buffer operators in too subjectively biased weighting the new or the old information in forecasting. These expand the application of the proposed method in time series analysis.

Keywords Time series, sequence operator, model stability, model perturbation analysis.

1 Introduction

Time series prediction models play important roles in many real-world applications, including economic and social forecasting problems[1, 2]. However, data noise and missing data are often encountered in model applications and they affect the effectiveness of prediction results.

LI Chong
School of Air Traffic Management, Civil Aviation Flight University of China, Guanghan 618307, *China. Email: lichong2007@126.com*
YANG Yingjie
*School of Computer Science and Informatics, De Montfort University, Leicester LE*19*BH, UK. Email: yyang@dmu.ac.uk*
ZHU Xinping (Corresponding author)
School of Air Traffic Management, Civil Aviation Flight University of China, Guanghan 618307, *China. Email: fdskkj@126.com*
* This research was supported by the National Key Research and Development Program of China under Grant No. 2021YFB2601704, the Science and Technology Project of Sichuan Province under Grant No. 2021YFS0319 and the Fundamental Research Funds for the Central Universities of China under Grant Nos. ZHMH2022-008 and J2022-062.
⋄

The existence and stability of stationary solutions is one of the basic problems of the models subject to exogenous random perturbations[3, 4]. Researchers have proposed ways and methods to improve the accuracy of model solutions[5, 6].

Sequence operators play an important role in reducing the interference information in collected data and highlighting the potential development process of analyzed objects. Among them are the widely used exponential smoothing operators[7], especially the seasonal exponential smoothing method in application of forecasting[8, 9]. The buffer operators in the grey systems theory also exhibit similar features[10]. They are based on the three axioms of buffer operator and suitable for small sample analysis. They greatly improve the performance of grey prediction models in real applications[11]. These operators can be further subdivided into weakening operators and strengthening operators[12, 13]. When it comes to the system development trend analysis or the series prediction, the weakening buffer operators (WBO) is preferred. These kind of operators include the average weakening buffer operator (AWBO), the geometric average weakening buffer operator (GAWBO), and the weighted WBOs (WAWBO,WGAWBO), etc.[14, 15] The main disadvantage of these operators is their subjective determination of the weight coefficients of data[16]. This will undoubtedly miss some useful information under certain conditions and limit their applications.

Wu et al.[17–19] first studied the essence of WBOs in grey prediction models. Based on the perturbation theory, they found that series prediction models are more sensitive to earlier data than newer data in a given sequence. Their findings reveal that the classic integral accumulated generating operator 1-AGO and extended fractional accumulated generating operator attach more importance to the earlier data than to recent data, and both operators dont satisfy the priority theory of new information in grey systems theory. To solve this problem, they later proposed some reverse operators in Wu et al.[17] and Wu et al.[19], which attach more importance to the more recent data and thus are consistent with the priority theory of new information. Recent studies show that these new operators could improve the prediction performance of time series models[20–22]. However, these new operators emphasize either the new data or the old data, and thus are only suitable for series analysis with special time preferences.

To solve the above problems, this paper applies a data preprocessing method based on a novel fractional bidirectional weakening buffer operator which is first proposed by Li et al.[23]. Recent studies [23, 24] have showed the good performance of this weakening buffer operator in prediction against data noise interference. The effect of this adopted operator, together with two other widely used operators, on improving the stability of time series models are examined and compared. Applying the perturbation theory, this study demonstrates the effectiveness of the fractional bidirectional weakening buffer operator in improving the stability of time series models. The proposed operator-based data preprocessing method enlarges the application of series prediction models by taking into account more objectively both the old and the new data in samples.

The paper is structured as follows: Section 2 introduces three important fractional weakening buffer operators used in series models. Section 3 analyzes the performance of the proposed operator-based data preprocessing method on the sensitivity of a fractional order accumulate discrete grey model $DGM^p(1,1)$. Section 4 shows the model perturbation analysis result of the proposed method and its comparison operator on another model − the new information prior grey model $NIGM(p,1)$. Case studies in Section 5 demonstrate the effect of the new method in improving model stability. Finally, conclusions are drawn in Section 6.

2 Fractional weakening buffer operators

Weakening buffer operators are commonly used in series prediction models for finding the implicit pattern in samples. In order to show the advantage of the adopted fractional bidirectional weakening buffer operator in the proposed data preprocessing method, two other widely used classic weakening buffer operators are introduced in Definition 2.1 and Definition 2.2. They are chosen as the comparative objects of the proposed one.

Definition 2.1 (see [18]) Let $X^{(0)} = \{x^{(0)}(1), x^{(0)}(2), ..., x^{(0)}(n)\}$ be the original sequence, and $D^pX^{(0)} = X^p = \{x^p(1), x^p(2), \ldots, x^p(n)\}$ be the fractional p-order$(0 < p < 1)$ accumulated generating operator sequence of non-negative $X^{(0)}$, with

$$x^p(k) = \sum_{i=1}^{k} C_{k-i+p-1}^{k-i} x^{(0)}(i), \quad (k = 1, 2, ...n) \tag{1}$$

where $C_{p-1}^0 = 1$, $C_{k-1}^k = 0$, $C_{k-i+p-1}^{k-i} = \frac{(k-i+p-1)(k-i+p-2)...(p+1)(p)}{(k-i)!}$.

Definition 2.2 (see [19]) Let $X^{(0)} = \{x^{(0)}(1),\ x^{(0)}(2), ..., x^{(0)}(n)\}$ be the original sequence, and $D^rX^{(0)} = X_{(r)} = \{x_{(r)}(1), x_{(r)}(2), ..., x_{(r)}(n)\}$ be defined as the reverse fractional r-order accumulated generating operator sequence of non-negative $X^{(0)}$, with operator D^r satisfying:

$$x_{(r)}(k) = \sum_{i=k}^{n} C_{i-k+r-1}^{i-k} x^{(0)}(i), \quad (k = 1, 2, ...n) \tag{2}$$

where $C_{r-1}^0 = 1$, $C_{n-1}^n = 0$, $C_{i-k+r-1}^{i-k} = \frac{(i-k+r-1)(i-k+r-2)\cdots(r+1)r}{(i-k)!}$.

If order parameters p in Definition 2.1 and r in Definition 2.2 are limited to positive integers, the p-AGO and the r-IAGO are obtained. Important properties of these operators have been intensively studied[19, 25]. In this study, the fractional bidirectional weakening buffer operator adopted in the proposed data preprocessing method of time series prediction models is given in the following definition:

Definition 2.3 (see [23]) For the original sequence $X^{(0)} = \{x^{(0)}(1), x^{(0)}(2), ..., x^{(0)}(n)\}$, an operator sequence $X_v = \{x_v(1), x_v(2), ..., x_v(n)\}$ is obtained by $D_vX^{(0)} = X_v$, where time series operator D_v $(v \in R^+)$ is a fractional bidirectional weakening buffer operator, and elements in sequence X_v satisfy:

$$x_v(i) = \frac{\sum\limits_{j=i-\alpha(i)}^{i} w(i-j)x^{(0)}(j) + \sum\limits_{j=i+1}^{i+\alpha(i)} w(j-i)x^{(0)}(j)}{\sum\limits_{j=i-\alpha(i)}^{i} w(i-j) + \sum\limits_{j=i+1}^{i+\alpha(i)} w(j-i)}, \qquad i = 1, 2, ...n. \tag{3}$$

where $\alpha(i) = \min(i-1, n-i)$, and

$$w(i) = \begin{cases} \left(n^v(v+1-n) + (n-1)^{v+1}\right)/\Gamma(v+2), & i = n \\ \left((i+1)^{v+1} - 2i^{v+1} + (i-1)^{v+1}\right)/\Gamma(v+2), & i = 1, 2, ...(n-1) \\ 1/\Gamma(v+2), & i = 0 \end{cases} \quad (4)$$

It has been proved in [23] that the fractional bidirectional weakening buffer operator D_v defined in equation (3) is a weakening buffer operator. According to Definition 2.3, value adjustment of an element in a given series is determined by both the older and the newer data, while it is exclusively determined by the old or the new data if applying Definition 2.1 or Definition 2.2. In the following sections, we will test the performance of operator D_v in improving the stability of two grey prediction models, which are based on the operators presented in Definition 2.1 and Definition 2.2, respectively.

3 Perturbation analysis of model 1: the fractional order accumulate discrete grey model $DGM^p(1,1)$ in [18]

Whether a time series model is sensitive to data samples plays different functions in different applications. In this section, perturbation theory is employed to test the effect of the fractional bidirectional weakening buffer operator D_v on the sensitivity of the fractional order accumulate discrete grey model $DGM^p(1,1)$ designed in [18].

Lemma 3.1 (see [26]) *Let matrixes $A \in C^{n\times n}, F \in C^{n\times n}$, vectors $b \in C^n, c \in C^n$. Assume that matrix or vector norm $\|.\|$ is tolerant, $rank(A) = rank(A+F) = n$ and matrix norm $\left\|A^{-1}\right\| \|F\| < 1$ is true, then the least squares estimation solution x of linear system models $Ax = b$ and the least squares estimation solution $x+h$ of model $(A+F)(x+h) = b+c$ satisfy:*

$$\|h\| \leq \frac{\kappa_\dagger}{\gamma_\dagger}\left(\frac{\|F\|_2\|x\|}{\|A\|} + \frac{\|c\|}{\|A\|} + \frac{\kappa_\dagger}{\gamma_\dagger}\frac{\|F\|_2\|\gamma_x\|}{\|A\|^2}\right) \quad (5)$$

where $\kappa_\dagger = ||A^{-1}||_2||A||$, $\gamma_\dagger = 1 - |A^{-1}||_2||F||_2$, $\gamma_x = b - Ax$.

To keep the consistency of parameters, the fractional order parameter p/q used in [18] is replaced by parameter p in the rest of this paper.

Definition 3.2 (see [18]) Let $X^{(0)} = \{x^{(0)}(1), x^{(0)}(2), ..., x^{(0)}(n)\}$ be the original sequence, and $X^p = \{x^p(1), x^p(2), \dots, x^p(n)\}$ be its fractional p-order accumulated generating operator sequence based on Definition 2.1, then the following time series model is called a fractional order accumulate discrete grey model $DGM^p(1,1)$:

$$x^p(k+1) = \beta_1 x^p(k) + \beta_2, \quad k = 1, 2, ...n-1 \quad (6)$$

The least squares estimate of model parameters β_1 and β_2 in model (6) are:

$$\begin{bmatrix} \beta_2 \\ \beta_1 \end{bmatrix} = (B^T B)^{-1} B^T Y \quad (7)$$

where

$$B = \begin{bmatrix} 1 & x^p(1) \\ 1 & x^p(2) \\ \vdots & \vdots \\ 1 & x^p(n-2) \\ 1 & x^p(n-1) \end{bmatrix}, \qquad Y = \begin{bmatrix} x^p(2) \\ x^p(3) \\ \vdots \\ x^p(n-1) \\ x^p(n) \end{bmatrix}$$

For simplicity, let $\beta = [\beta_2, \beta_1]^T$. Perturbation analysis result of model (6) in [18] is summarized in following Theorems 3.3 to 3.5.

Theorem 3.3 (see [18]) *For the $DGM^p(1,1)$ model with original data $X^{(0)}=\{x^{(0)}(1), ..., x^{(0)}(n)\}$, let β be its least squares estimation. When the raw data is disturbed, $DGM^p(1,1)$ becomes $(B+\Delta B)\widetilde{\beta} = (Y + \Delta Y)$, where ΔB and ΔY are determined by the disturbance item ε. Assume that the matrix norm condition $||B^{-1}||_2||\Delta B||_2 < 1$ hold for $DGM^p(1,1)$. Let $\kappa_\dagger = ||B^{-1}||_2||B||$, $\gamma_\dagger = 1 - ||B^{-1}||_2||\Delta B||_2$, $\gamma_\beta = Y - B\beta$. If the disturbance occurs in the first data of the original sequence, that is $\widetilde{x}^{(0)}(1) = x^{(0)}(1) + \varepsilon$, then the difference between solutions $\widetilde{\beta}$ and β satisfies:*

$$\left\|\delta(x^{(0)}(1))\right\| \leq |\varepsilon| \frac{\kappa_\dagger}{\gamma_\dagger} \left(\frac{\sqrt{\sum\limits_{k=1}^{n-1}\left(C_{k+p-2}^{k-1}\right)^2}\,\|\beta\|}{\|B\|} + \frac{\sqrt{\sum\limits_{k=2}^{n}\left(C_{k+p-2}^{k-1}\right)^2}}{\|B\|} + \frac{\kappa_\dagger}{\gamma_\dagger}\frac{\sqrt{\sum\limits_{k=1}^{n-1}\left(C_{k+p-2}^{k-1}\right)^2}\,\|\gamma_\beta\|}{\|B\|^2} \right). \tag{8}$$

Theorem 3.4 (see [18]) *For the same model and parameters as those in Theorem 3.3, if the disturbance occurs in the non-boundary nodes, that is $\widetilde{x}^{(0)}(k) = x^{(0)}(k) + \varepsilon, (k=2,3,...n-1)$, then the difference between solutions of model $DGM^p(1,1)$ satisfies:*

$$\left\|\delta(x^{(0)}(k))\right\| \leq |\varepsilon| \frac{\kappa_\dagger}{\gamma_\dagger} \left(\frac{\sqrt{\sum\limits_{i=1}^{n-k}\left(C_{i+p-2}^{i-1}\right)^2}\,\|\beta\|}{\|B\|} + \frac{\sqrt{\sum\limits_{i=1}^{n-k+1}\left(C_{i+p-2}^{i-1}\right)^2}}{\|B\|} + \frac{\kappa_\dagger}{\gamma_\dagger}\frac{\sqrt{\sum\limits_{i=1}^{n-k}\left(C_{i+p-2}^{i-1}\right)^2}\,\|\gamma_\beta\|}{\|B\|^2} \right). \tag{9}$$

Theorem 3.5 (see [18]) *For the same model and parameters as those in Theorem 3.3, if the disturbance occurs in the end node of the original sequence, that is $\widetilde{x}^{(0)}(n) = x^{(0)}(n) + \varepsilon$, then the difference between solutions of model $DGM^p(1,1)$ satisfies:*

$$\left\|\delta(x^{(0)}(n))\right\| \leq \frac{\kappa_\dagger}{\gamma_\dagger}\frac{|\varepsilon|}{\|B\|} \tag{10}$$

Based on above theorems, Wu et al.[18] demonstrates that the proposed $DGM^p(1,1)$ has better solution stability than the classic $GM(1,1)$ model. Now, to test the advantage of the fractional bidirectional weakening buffer operator D_v in improving model stability, we apply a first preprocessing step to the original time series based on operator D_v, then carry out the

$DGM^p(1,1)$ analysis. Perturbation theory is employed to test the effect of operator D_v on the solution stability of $DGM^p(1,1)$.

Let $X^{(0)} = \{x^{(0)}(1), x^{(0)}(2), ..., x^{(0)}(n)\}$ be the original sequence, $X_v = \{x_v(1), x_v(2), ..., x_v(n)\}$ be its weakening buffer operator sequence based on operator D_v. Apply $DGM^p(1,1)$ model to sequence X_v, let β_v be the new model solution values of parameters $[\beta_2, \beta_1]^T$. Then we have:

$$\beta_v = ({B_v}^T B_v)^{-1} {B_v}^T Y_v \tag{11}$$

where $x_v^p(k) = \sum_{i=1}^k C_{k-i+p-1}^{k-i} x_v(i)$, and

$$B_v = \begin{bmatrix} 1 & x_v^p(1) \\ 1 & x_v^p(2) \\ \vdots & \vdots \\ 1 & x_v^p(n-2) \\ 1 & x_v^p(n-1) \end{bmatrix}, \qquad Y_v = \begin{bmatrix} x_v^p(2) \\ x_v^p(3) \\ \vdots \\ x_v^p(n-1) \\ x_v^p(n) \end{bmatrix}$$

.

When the raw data is disturbed, let ΔB_v and ΔY_v be the disturbance items of B_v and Y_v, respectively, and $\widetilde{\beta}_v$ be the new model solution. Assume that $||B_v^{-1}||_2 ||\Delta B_v||_2 < 1$ and set $\kappa_{v\dagger} = ||B_v^{-1}||_2 ||B_v||$, $\gamma_{v\dagger} = 1 - ||B_v^{-1}||_2 ||\Delta B_v||_2$ and $\gamma_{v\beta} = Y_v - B_v\beta_v$. Parameters $\alpha(i)$ and $w(i)$ used in the rest part of this section are same as those defined in model (3). The main results are stated in the following theorems.

Theorem 3.6 *If raw data disturbance occurs in the first data of the original sequence, that is $\widetilde{x}^{(0)}(1) = x^{(0)}(1) + \varepsilon$, then the difference between $\widetilde{\beta}_v$ and β_v satisfies:*

$$\left\|\delta(x^{(0)}(1))\right\| \le |\varepsilon| \frac{\kappa_{v\dagger}}{\gamma_{v\dagger}} \left(\frac{M_v \|\beta_v\|}{\|B_v\|} + \frac{N_v}{\|B_v\|} + \frac{\kappa_{v\dagger}}{\gamma_{v\dagger}} \frac{M_v \|\gamma_{v\beta}\|}{\|B_v\|^2} \right), \tag{12}$$

where: (i). if sequence size n is even, then:

$$M_v = \sqrt{\sum_{i=1}^{n/2} \left(\sum_{j=1}^{i} C_{j+p-2}^{j-1} \frac{w(i-j)}{w(0)+2\sum\limits_{h=1}^{\alpha(i-j+1)} w(h)} \right)^2 + \sum_{i=n/2+1}^{n-1} \left(\sum_{j=1}^{n/2} C_{i+p-j-1}^{i-j} \frac{w(j-1)}{w(0)+2\sum\limits_{h=1}^{\alpha(j)} w(h)} \right)^2},$$

$$N_v = \sqrt{\sum_{i=2}^{n/2} \left(\sum_{j=1}^{i} C_{j+p-2}^{j-1} \frac{w(i-j)}{w(0)+2\sum\limits_{h=1}^{\alpha(i-j+1)} w(h)} \right)^2 + \sum_{i=n/2+1}^{n} \left(\sum_{j=1}^{n/2} C_{i+p-j-1}^{i-j} \frac{w(j-1)}{w(0)+2\sum\limits_{h=1}^{\alpha(j)} w(h)} \right)^2}. \tag{13}$$

(ii). if sequence size n is odd, then:

$$M_v=\sqrt{\sum_{i=1}^{(n+1)/2}\left(\sum_{j=1}^{i}C_{j+p-2}^{j-1}\frac{w(i-j)}{w(0)+2\sum\limits_{h=1}^{\alpha(i-j+1)}w(h)}\right)^2+\sum_{i=(n+3)/2}^{n-1}\left(\sum_{j=1}^{(n+1)/2}C_{i+p-j-1}^{i-j}\frac{w(j-1)}{w(0)+2\sum\limits_{h=1}^{\alpha(j)}w(h)}\right)^2},$$

$$N_v=\sqrt{\sum_{i=2}^{(n+1)/2}\left(\sum_{j=1}^{i}C_{j+p-2}^{j-1}\frac{w(i-j)}{w(0)+2\sum\limits_{h=1}^{\alpha(i-j+1)}w(h)}\right)^2+\sum_{i=(n+3)/2}^{n}\left(\sum_{j=1}^{(n+1)/2}C_{i+p-j-1}^{i-j}\frac{w(j-1)}{w(0)+2\sum\limits_{h=1}^{\alpha(j)}w(h)}\right)^2}. \tag{14}$$

Proof (i). Sequence size n is even. When $\widetilde{x}^{(0)}(1)=x^{(0)}(1)+\varepsilon$, according to the logic of $DGM^p(1,1)$, we have:

$$\Delta B_v=\begin{bmatrix}
0 & \varepsilon\\
0 & C_p^1\varepsilon+\frac{w(1)\varepsilon}{w(0)+2w(1)}\\
\vdots & \vdots\\
0 & C_{p+n/2-2}^{n/2-1}\varepsilon+C_{p+n/2-3}^{n/2-2}\frac{w(1)\varepsilon}{w(0)+2w(1)}+\cdots\frac{w(n/2-1)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2)}w(h)}\\
0 & C_{p+n/2-1}^{n/2}\varepsilon+C_{p+n/2-2}^{n/2-1}\frac{w(1)\varepsilon}{w(0)+2w(1)}+\cdots C_p^1\frac{w(n/2-1)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2)}w(h)}\\
0 & C_{p+n/2}^{n/2+1}\varepsilon+C_{p+n/2-1}^{n/2}\frac{w(1)\varepsilon}{w(0)+2w(1)}+\cdots C_{p+1}^2\frac{w(n/2-1)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2)}w(h)}\\
\vdots & \vdots\\
0 & C_{p+n-3}^{n-2}\varepsilon+C_{p+n-4}^{n-3}\frac{w(1)\varepsilon}{w(0)+2w(1)}+\cdots C_{p+n/2-2}^{n/2-1}\frac{w(n/2-1)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2)}w(h)}
\end{bmatrix},$$

$$\Delta Y_v=\begin{bmatrix}
C_p^1\varepsilon+\frac{w(1)\varepsilon}{w(0)+2w(1)}\\
C_{p+1}^2\varepsilon+C_p^1\frac{w(1)\varepsilon}{w(0)+2w(1)}+\frac{w(2)\varepsilon}{w(0)+2(w(1)+w(2))}\\
\vdots\\
C_{p+n/2-1}^{n/2}\varepsilon+C_{p+n/2-2}^{n/2-1}\frac{w(1)\varepsilon}{w(0)+2w(1)}+\cdots C_p^1\frac{w(n/2-1)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2)}w(h)}\\
C_{p+n/2}^{n/2+1}\varepsilon+C_{p+n/2-1}^{n/2}\frac{w(1)\varepsilon}{w(0)+2w(1)}+\cdots C_{p+1}^2\frac{w(n/2-1)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2)}w(h)}\\
C_{p+n/2+1}^{n/2+2}\varepsilon+C_{p+n/2}^{n/2+1}\frac{w(1)\varepsilon}{w(0)+2w(1)}+\cdots C_{p+2}^3\frac{w(n/2-1)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2)}w(h)}\\
\vdots\\
C_{p+n-2}^{n-1}\varepsilon+C_{p+n-3}^{n-2}\frac{w(1)\varepsilon}{w(0)+2w(1)}+\cdots C_{p+n/2-1}^{n/2}\frac{w(n/2-1)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2)}w(h)}
\end{bmatrix}.$$

Now calculate the norms of matrixes ΔB_v and ΔY_v, extract parameter ε and apply Lemma 3.1, parameter expressions in equation (13) are obtained.

(ii). When sequence size n is odd, we have:

$$\Delta B_v=\begin{bmatrix} 0 & \varepsilon \\ 0 & C_p^1\varepsilon+\frac{w(1)\varepsilon}{w(0)+2w(1)} \\ \vdots & \vdots \\ 0 & C_{p+(n-3)/2}^{(n-1)/2}\varepsilon+C_{p+(n-5)/2}^{(n-3)/2}\frac{w(1)\varepsilon}{w(0)+2w(1)}+\cdots\frac{w((n-1)/2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)}w(h)} \\ 0 & C_{p+(n-1)/2}^{(n+1)/2}\varepsilon+C_{p+(n-3)/2}^{(n-1)/2}\frac{w(1)\varepsilon}{w(0)+2w(1)}+\cdots C_p^1\frac{w((n-1)/2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)}w(h)} \\ \vdots & \vdots \\ 0 & C_{p+n-3}^{n-2}\varepsilon+C_{p+n-4}^{n-3}\frac{w(1)\varepsilon}{w(0)+2w(1)}+\cdots C_{p+(n-5)/2}^{(n-3)/2}\frac{w((n-1)/2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)}w(h)} \end{bmatrix},$$

$$\Delta Y_v=\begin{bmatrix} C_p^1\varepsilon+\frac{w(1)\varepsilon}{w(0)+2w(1)} \\ C_{p+1}^2\varepsilon+C_p^1\frac{w(1)\varepsilon}{w(0)+2w(1)}+\frac{w(2)\varepsilon}{w(0)+2w(1)+2w(2)} \\ \vdots \\ C_{p+(n-1)/2}^{(n+1)/2}\varepsilon+C_{p+(n-3)/2}^{(n-1)/2}\frac{w(1)\varepsilon}{w(0)+2w(1)}+\cdots C_p^1\frac{w((n-1)/2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)}w(h)} \\ C_{p+(n+1)/2}^{(n+3)/2}\varepsilon+C_{p+(n-1)/2}^{(n+1)/2}\frac{w(1)\varepsilon}{w(0)+2w(1)}+\cdots C_{p+1}^2\frac{w((n-1)/2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)}w(h)} \\ \vdots \\ C_{p+n-2}^{n-1}\varepsilon+C_{p+n-3}^{n-2}\frac{w(1)\varepsilon}{w(0)+2w(1)}+\cdots C_{p+(n-3)/2}^{(n-1)/2}\frac{w((n-1)/2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)}w(h)} \end{bmatrix}.$$

Then calculate matrix norm $||\Delta B_v||_2$ and vector norm $||\Delta Y_v||$, extract parameter ε and apply Lemma 3.1, parameter expressions in equation (14) are obtained. ■

Theorem 3.7 *If raw data disturbance occurs in the non-boundary nodes of the first half of the original sequence, that is $\widetilde{x}^{(0)}(k)=x^{(0)}(k)+\varepsilon$, then the difference between $\widetilde{\beta}_v$ and β_v satisfies expression (12), but the values of parameters M_v and N_v are determined by the following scenarios. Let φ_1 be the largest integer less than $k/2$, φ_2 be the largest integer less than $(k-1)/2$, set $c_1=j-1$, $c_2=j+p-2$, $c_3=i+j-\varphi_1-1-n/2$, $c_4=p+i+j-\varphi_1-2-n/2$, $c_5=i+j-\varphi_2-1-(n+1)/2$, $c_6=p+i+j-\varphi_2-2-(n+1)/2$, then M_v and N_v are:*

(i). if sequence size n is even, then $k = 2, 3, \dots n/2$ and

$$M_v = \sqrt{\sum_{i=\varphi_1+1}^{n/2+\varphi_1}\left(\sum_{j=1}^{i-\varphi_1} C_{c_2}^{c_1}\frac{w(|k+j-i-1|)}{w(0)+2\sum\limits_{h=1}^{\alpha(i-j+1)} w(h)}\right)^2 + \sum_{i=n/2+\varphi_1+1}^{n-1}\left(\sum_{j=1}^{n/2} C_{c_4}^{c_3}\frac{w(|k+j-\varphi_1-1-n/2|)}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+\varphi_1+1-j)} w(h)}\right)^2},$$

$$N_v = \sqrt{(M_v)^2 + \left(\sum_{j=1}^{n/2} C_{p+j-\varphi_1-2+n/2}^{j-\varphi_1-1+n/2}\frac{w(|k+j-\varphi_1-1-n/2|)}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+\varphi_1+1-j)} w(h)}\right)^2}. \tag{15}$$

(ii). if sequence size n is odd, then $k = 2, 3, \dots (n+1)/2$ and

$$M_v = \sqrt{\sum_{i=\varphi_1+1}^{(n+1)/2+\varphi_2}\left(\sum_{j=1}^{i-\varphi_1} C_{c_2}^{c_1}\frac{w(|k+j-i-1|)}{w(0)+2\sum\limits_{h=1}^{\alpha(i-j+1)} w(h)}\right)^2 + \sum_{i=(n+1)/2+\varphi_2+1}^{n-1}\left(\sum_{j=1}^{(n+1)/2+\varphi_2-\varphi_1} C_{c_6}^{c_5}\frac{w(|k+j-\varphi_2-1-(n+1)/2|)}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2+\varphi_2+1-j)} w(h)}\right)^2},$$

$$N_v = \sqrt{(M_v)^2 + \left(\sum_{j=1}^{(n+1)/2+\varphi_2-\varphi_1} C_{p+j-\varphi_2-2+(n-1)/2}^{j-\varphi_2-1+(n-1)/2}\frac{w(|k+j-\varphi_2-1-(n+1)/2|)}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2+\varphi_2+1-j)} w(h)}\right)^2}. \tag{16}$$

Proof Please see the Appendix A.1.

Theorem 3.8 *If raw data disturbance occurs in the second half of the original sequence, excluding the last point, then the difference between $\widetilde{\beta}_v$ and β_v satisfies expression (12). Let φ_3 be the largest integer less than $(n-k)/2$, φ_4 be the largest integer less than $(n-k+1)/2$, c_1 and c_2 are same as those defined in Theorem 3.7, set $c_7 = i+j+\varphi_4-1-n$, $c_8 = p+i+j+\varphi_4-2-n$, then M_v and N_v are:*

(i). if sequence size n is even, then $k = n/2+1, n/2+2, \dots n-1$ and

$$M_v = \sqrt{\sum_{i=n/2+1-\varphi_4}^{n-\varphi_4}\left(\sum_{j=1}^{i-n/2+\varphi_4} C_{c_2}^{c_1}\frac{w(|k+j-i-1|)}{w(0)+2\sum\limits_{h=1}^{\alpha(i-j+1)} w(h)}\right)^2 + \sum_{i=n+1-\varphi_4}^{n-1}\left(\sum_{j=1}^{n/2} C_{c_8}^{c_7}\frac{w(|k+j+\varphi_4-1-n|)}{w(0)+2\sum\limits_{h=1}^{\alpha(n+1-\varphi_4-j)} w(h)}\right)^2},$$

$$N_v = \sqrt{(M_v)^2 + \left(\sum_{j=1}^{n/2} C_{p+j+\varphi_4-2}^{j+\varphi_4-1}\frac{w(|k+j+\varphi_4-1-n|)}{w(0)+2\sum\limits_{h=1}^{\alpha(n+1-\varphi_4-j)} w(h)}\right)^2}. \tag{17}$$

(ii). if sequence size n is odd, then $k=(n+1)/2+1,(n+1)/2+2,\dots n-1$ and

$$M_v=\sqrt{\sum_{i=(n+1)/2-\varphi_3}^{n-\varphi_4}\left(\sum_{j=1}^{i+\varphi_3-(n-1)/2}C_{c_2}^{c_1}\frac{w(|k+j-i-1|)}{w(0)+2\sum\limits_{h=1}^{\alpha(i+1-j)}w(h)}\right)^2+\sum_{i=n-\varphi_4+1}^{n-1}\left(\sum_{j=1}^{(n+1)/2+\varphi_3-\varphi_4}C_{c_8}^{c_7}\frac{w(|k+j+\varphi_4-1-n|)}{w(0)+2\sum\limits_{h=1}^{\alpha(n+1-\varphi_4-j)}w(h)}\right)^2},$$

$$N_v=\sqrt{(M_v)^2+\left(\sum_{j=1}^{(n+1)/2+\varphi_3-\varphi_4}C_{p+j+\varphi_4-2}^{j+\varphi_4-1}\frac{w(|k+j+\varphi_4-1-n|)}{w(0)+2\sum\limits_{h=1}^{\alpha(n+1-\varphi_4-j)}w(h)}\right)^2}. \tag{18}$$

Proof Please see the Appendix A.2.

Theorem 3.9 *If raw data disturbance occurs in the last time point of the original sequence, that is $\widetilde{x}^{(0)}(n)=x^{(0)}(n)+\varepsilon$, then the difference between $\widetilde{\beta}_v$ and β_v satisfies expression (12), but the values of parameters M_v and N_v are determined by the size of samples.*

(i). if sequence size n is even, then

$$\begin{aligned}M_v&=\sqrt{\sum_{i=1}^{n/2-1}\left(\sum_{j=1}^{i}C_{i+p-j-1}^{i-j}\frac{w(n/2-j)}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+j)}w(h)}\right)^2},\\N_v&=\sqrt{\sum_{i=1}^{n/2}\left(\sum_{j=1}^{i}C_{i+p-j-1}^{i-j}\frac{w(n/2-j)}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+j)}w(h)}\right)^2}.\end{aligned} \tag{19}$$

(ii). if sequence size n is odd, then

$$\begin{aligned}M_v&=\sqrt{\sum_{i=1}^{(n-1)/2}\left(\sum_{j=1}^{i}C_{i+p-j-1}^{i-j}\frac{w((n+1)/2-j)}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2+j-1)}w(h)}\right)^2},\\N_v&=\sqrt{\sum_{i=1}^{(n+1)/2}\left(\sum_{j=1}^{i}C_{i+p-j-1}^{i-j}\frac{w((n+1)/2-j)}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2+j-1)}w(h)}\right)^2}.\end{aligned} \tag{20}$$

Proof Please see the Appendix A.3.

4 Perturbation analysis of model 2: the $NIGM(p, 1)$ in [19]

The fractional order accumulate discrete grey model in Section 3 emphasizes old data in modeling samples. To give more weight to new information, Wu et al.[19] provides the following model.

Definition 4.1 (see [19]) Let $X^{(0)} = \{x^{(0)}(1), x^{(0)}(2), ..., x^{(0)}(n)\}$ be the original sequence, $X_{(1-p)} = \{x_{(1-p)}(1), x_{(1-p)}(2), \ldots, x_{(1-p)}(n)\}(0 < p < 1)$ be the fractional $(1-p)$-order reverse accumulated generating operator sequence of $X^{(0)}$ (see Definition 2.2). Set mean formula $z^{(0)}(k) = \left(x^{(0)}(k) + x^{(0)}(k-1)\right)/2$ and formula $d_{(1)}x_{(1-p)}(k) = x_{(1-p)}(k) - x_{(1-p)}(k-1)$, then the new information prior grey model $NIGM(p, 1)$ is defined by:

$$d_{(1)}x_{(1-p)}(k) + \tau_1 z^{(0)}(k) = \tau_2 \tag{21}$$

The least squares estimation of model parameter τ_1 and τ_2 in (21) can be obtained by:

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = (E^T E)^{-1} E^T U \tag{22}$$

where

$$E = \begin{bmatrix} -z^{(0)}(2) & 1 \\ -z^{(0)}(3) & 1 \\ \vdots & \vdots \\ -z^{(0)}(n) & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} d_{(1)}x_{(1-p)}(2) \\ d_{(1)}x_{(1-p)}(3) \\ \vdots \\ d_{(1)}x_{(1-p)}(n) \end{bmatrix}.$$

Let $\tau = [\tau_1, \tau_2]^T$, ΔE and ΔU be the disturbance items of model parameters E and U respectively, ε be the disturbance item on raw data. Assume that the matrix norm condition $||E^{-1}||_2||\Delta E||_2 < 1$ holds for model $NIGM(p, 1)$. Let $\mu_\dagger = ||E^{-1}||_2||E||$, $\eta_\dagger = 1 - ||E^{-1}||_2||\Delta E||_2$, $\eta_\tau = U - E\tau$. Based on Lemma 3.1, the perturbation analysis results of $NIGM(p, 1)$ is summarized in following Theorems 4.2 to 4.4. Theorem 4.2 directly comes from Wu et al.[19] while Theorems 4.3 to 4.4 are derived from the corrected parameters.

Theorem 4.2 (see [19]) *If the disturbance ε occurs in the first data of the original sequence, that is $\widetilde{x}^{(0)}(1) = x^{(0)}(1) + \varepsilon$, then the difference between solutions $\widetilde{\tau}$ and τ satisfies:*

$$\left\|\delta(x^{(0)}(1))\right\| \le |\varepsilon| \frac{\mu_\dagger}{\eta_\dagger} \left(\frac{\|\tau\|}{2\,\|E\|} + \frac{1}{\|E\|} + \frac{\mu_\dagger}{\eta_\dagger} \frac{\|\eta_\tau\|}{2\|E\|^2} \right). \tag{23}$$

Theorem 4.3 *If the disturbance occurs in the non-boundary nodes, that is $\widetilde{x}^{(0)}(i) = x^{(0)}(i) + \varepsilon$, $(i = 2, 3, ..., n-1)$, then the difference between solutions of model $NIGM(p, 1)$ with and without data disturbance satisfies:*

$$\left\|\delta(x^{(0)}(i))\right\| \le |\varepsilon| \frac{\mu_\dagger}{\eta_\dagger} \left(\frac{\sqrt{2}\,\|\tau\|}{2\,\|E\|} + \frac{\sqrt{\sum_{j=2}^{i} \left(C_{j-2-p}^{j-1}\right)^2 + 1}}{\|E\|} + \frac{\mu_\dagger}{\eta_\dagger} \frac{\sqrt{2}\,\|\eta_\tau\|}{2\|E\|^2} \right). \tag{24}$$

Proof (i). When $\widetilde{x}^{(0)}(2) = x^{(0)}(2) + \varepsilon$, disturbance items of parameters ΔE and ΔU are:

$$\Delta E = \begin{bmatrix} -\varepsilon/2 & 0 \\ -\varepsilon/2 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \qquad \Delta U = \begin{bmatrix} p\varepsilon \\ -\varepsilon \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then $||\Delta E||_2 = \sqrt{2}|\varepsilon|/2$, $||\Delta U|| = |\varepsilon|\sqrt{p^2+1}$.

(ii). When $\widetilde{x}^{(0)}(3) = x^{(0)}(3) + \varepsilon$, disturbance items of model parameters ΔE and ΔU are:

$$\Delta E = \begin{bmatrix} 0 & 0 \\ -\varepsilon/2 & 0 \\ -\varepsilon/2 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \qquad \Delta U = \begin{bmatrix} -C_{1-p}^{2}\varepsilon \\ p\varepsilon \\ -\varepsilon \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then $||\Delta E||_2 = \sqrt{2}|\varepsilon|/2$, $||\Delta U|| = |\varepsilon|\sqrt{(p^4-2p^3+5p^2)/4+1}$.

Likewise, (iii). when $\widetilde{x}^{(0)}(i) = x^{(0)}(i) + \varepsilon$,$(i = 4, \ldots n-1)$, disturbance items of model parameters ΔE and ΔU are:

$$\Delta E = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ -\varepsilon/2 & 0 \\ -\varepsilon/2 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \qquad \Delta U = \begin{bmatrix} -C_{i-p-2}^{i-1}\varepsilon \\ -C_{i-p-3}^{i-2}\varepsilon \\ \vdots \\ p\varepsilon \\ -\varepsilon \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then $||\Delta E||_2 = \sqrt{2}|\varepsilon|/2$, $||\Delta U|| = |\varepsilon|\sqrt{\sum_{j=2}^{i}\left(C_{j-2-p}^{j-1}\right)^2+1}$.

Summarize the above three scenarios, perturbation result (24) is obtained. ■

Theorem 4.4 *If the disturbance occurs in the last sample point, that is* $\widetilde{x}^{(0)}(n) = x^{(0)}(n) + \varepsilon$, *then the difference between solutions of model* $NIGM(p,1)$ *satisfies:*

$$\left\|\delta(x^{(0)}(n))\right\| \le |\varepsilon|\,\frac{\mu_\dagger}{\eta_\dagger}\left(\frac{\|\tau\|}{2\,\|E\|} + \frac{\sqrt{\sum_{j=2}^{n}\left(C_{j-2-p}^{j-1}\right)^2}}{\|E\|} + \frac{\mu_\dagger}{\eta_\dagger}\,\frac{\|\eta_\tau\|}{2\|E\|^2}\right). \tag{25}$$

Proof When $\widetilde{x}^{(0)}(n) = x^{(0)}(n) + \varepsilon$, disturbance items of parameters ΔE and ΔU are:

$$\Delta E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ -\varepsilon/2 & 0 \end{bmatrix}, \qquad \Delta U = \begin{bmatrix} -C_{n-p-2}^{n-1}\varepsilon \\ -C_{n-p-3}^{n-2}\varepsilon \\ \vdots \\ -C_{1-p}^{2}\varepsilon \\ p\varepsilon \end{bmatrix}.$$

Then $||\Delta E||_2 = |\varepsilon|/2$, $||\Delta U|| = |\varepsilon|\sqrt{\sum_{j=2}^{n}\left(C_{j-2-p}^{j-1}\right)^2}$. Apply these norms into Lemma 3.1, perturbation bound (25) is proved. ■

Now we consider the effect of operator D_v on the stability of model $NIGM(p,1)$. $X^{(0)} = \{x^{(0)}(1), x^{(0)}(2), ..., x^{(0)}(n)\}$ and $X_v = \{x_v(1), x_v(2), ..., x_v(n)\}$ are the same as those defined in Section 3. Set $x_v^{(1-p)}(k) = \sum_{i=k}^{n} C_{i-k-p}^{i-k} x_v(i)$, $d_{(1)}x_v^{(1-p)}(k) = x_v^{(1-p)}(k) - x_v^{(1-p)}(k-1)$ and $z_v^{(0)}(k) = (x_v(k) + x_v(k-1))/2$. Let τ_v be the least squares solution of model $NIGM(p,1)$ based on sequence X_v, then we have:

$$\tau_v = \left(E_v^T E_v\right)^{-1} E_v^T U_v \tag{26}$$

where

$$E_v = \begin{bmatrix} -z_v^{(0)}(2) & 1 \\ -z_v^{(0)}(3) & 1 \\ \vdots & \vdots \\ -z_v^{(0)}(n) & 1 \end{bmatrix}, \qquad U_v = \begin{bmatrix} d_{(1)}x_v^{(1-p)}(2) \\ d_{(1)}x_v^{(1-p)}(3) \\ \vdots \\ d_{(1)}x_v^{(1-p)}(n) \end{bmatrix}.$$

Let ΔE_v and ΔU_v be the disturbance items of E_v and U_v, respectively, and $\widetilde{\tau}_v$ be the new model solution. Assume that $||E_v^{-1}||_2||\Delta E_v||_2 < 1$ holds and let $\eta_{v\dagger} = 1 - ||E_v^{-1}||_2||\Delta E_v||_2$, $\mu_{v\dagger} = ||E_v^{-1}||_2||E_v||$, $\eta_{v\tau} = U_v - E_v\tau_v$. Perturbation analysis results of model $NIGM(p,1)$ with operator D_v are stated in the following theorems.

Theorem 4.5 *If raw data disturbance occurs in the first data of the original sequence, that is $\widetilde{x}^{(0)}(1) = x^{(0)}(1) + \varepsilon$, then the difference between $\widetilde{\tau}_v$ and τ_v satisfies:*

$$\left\|\delta(x^{(0)}(1))\right\| \le |\varepsilon| \frac{\mu_{v\dagger}}{\eta_{v\dagger}} \left(\frac{P_v\|\tau_v\|}{\|E_v\|} + \frac{Q_v}{\|E_v\|} + \frac{\mu_{v\dagger}}{\eta_{v\dagger}} \frac{P_v\|\eta_{v\tau}\|}{\|E_v\|^2} \right) \tag{27}$$

where (i). if sequence size n is even, then:

$$P_v = \frac{1}{2}\sqrt{\sum_{i=1}^{n/2-1}\left(\frac{w(i-1)}{w(0)+2\sum\limits_{k=1}^{\alpha(i)} w(k)} + \frac{w(i)}{w(0)+2\sum\limits_{k=1}^{\alpha(i+1)} w(k)}\right)^2 + \left(\frac{w(n/2-1)}{w(0)+2\sum\limits_{k=1}^{\alpha(n/2)} w(k)}\right)^2},$$

$$Q_v = \sqrt{\sum_{i=1}^{n/2-1}\left(\frac{w(i-1)}{w(0)+2\sum\limits_{k=1}^{\alpha(i)} w(k)} + \sum_{j=1}^{n/2-i} C_{j-p-1}^{j}\frac{w(i+j-1)}{w(0)+2\sum\limits_{k=1}^{\alpha(i+j)} w(k)}\right)^2 + \left(\frac{w(n/2-1)}{w(0)+2\sum\limits_{k=1}^{\alpha(n/2)} w(k)}\right)^2}. \tag{28}$$

(ii). if sequence size n is odd, then:

$$P_v = \frac{1}{2}\sqrt{\sum_{i=1}^{(n-1)/2}\left(\frac{w(i-1)}{w(0)+2\sum\limits_{k=1}^{\alpha(i)} w(k)} + \frac{w(i)}{w(0)+2\sum\limits_{k=1}^{\alpha(i+1)} w(k)}\right)^2 + \left(\frac{w((n-1)/2)}{w(0)+2\sum\limits_{k=1}^{\alpha((n+1)/2)} w(k)}\right)^2},$$

$$Q_v = \sqrt{\sum_{i=1}^{(n-1)/2}\left(\frac{w(i-1)}{w(0)+2\sum\limits_{k=1}^{\alpha(i)} w(k)} + \sum_{j=1}^{(n+1)/2-i} C_{j-p-1}^{j}\frac{w(i+j-1)}{w(0)+2\sum\limits_{k=1}^{\alpha(i+j)} w(k)}\right)^2 + \left(\frac{w((n-1)/2)}{w(0)+2\sum\limits_{k=1}^{\alpha((n+1)/2)} w(k)}\right)^2}. \tag{29}$$

Proof (i). Sequence size n is even. When $\widetilde{x}^{(0)}(1) = x^{(0)}(1) + \varepsilon$, according to the logic of $NIGM(p,1)$, we have:

$$\Delta E_v = \begin{bmatrix} -\frac{\varepsilon}{2}\left(1+\frac{w(1)}{w(0)+2w(1)}\right) & 0 \\ -\frac{\varepsilon}{2}\left(\frac{w(1)}{w(0)+2w(1)} + \frac{w(2)}{w(0)+2\sum\limits_{h=1}^{\alpha(3)} w(h)}\right) & 0 \\ \vdots & \vdots \\ -\frac{\varepsilon}{2}\left(\frac{w(n/2-2)}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2-1)} w(h)} + \frac{w(n/2-1)}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2)} w(h)}\right) & 0 \\ -\frac{\varepsilon}{2}\frac{w(n/2-1)}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2)} w(h)} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \Delta U_v = \begin{bmatrix} -\varepsilon-\varepsilon\sum\limits_{i=1}^{n/2-1}\left(C_{i-p-1}^{i}\right)\frac{w(i)}{w(0)+2\sum\limits_{h=1}^{\alpha(i+1)} w(h)} \\ -\varepsilon\frac{w(1)}{w(0)+2w(1)} - \varepsilon\sum\limits_{i=1}^{n/2-2}\left(C_{i-p-1}^{i}\right)\frac{w(i+1)}{w(0)+2\sum\limits_{h=1}^{\alpha(i+2)} w(h)} \\ \vdots \\ -\varepsilon\frac{w(n/2-2)}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2-1)} w(h)} + \varepsilon p\frac{w(n/2-1)}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2)} w(h)} \\ -\varepsilon\frac{w(n/2-1)}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2)} w(h)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then calculate matrix norm $||\Delta E_v||_2$ and vector norm $||\Delta U_v||$, extract parameter ε and apply Lemma 3.1, parameter expressions in equation (28) are obtained.

(ii). When sequence size n is odd, we obtain:

$$\Delta E_v=\begin{bmatrix} -\frac{\varepsilon}{2}\left(1+\frac{w(1)}{w(0)+2w(1)}\right) & 0\\ -\frac{\varepsilon}{2}\left(\frac{w(1)}{w(0)+2w(1)}+\frac{w(2)}{w(0)+2\sum\limits_{h=1}^{\alpha(3)}w(h)}\right) & 0\\ \vdots & \vdots\\ -\frac{\varepsilon}{2}\left(\frac{w((n-3)/2)}{w(0)+2\sum\limits_{h=1}^{\alpha((n-1)/2)}w(h)}+\frac{w((n-1)/2)}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)}w(h)}\right) & 0\\ -\frac{\varepsilon}{2}\frac{w((n-1)/2)}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)}w(h)} & 0\\ 0 & 0\\ \vdots & \vdots\\ 0 & 0 \end{bmatrix},\ \Delta U_v=\begin{bmatrix} -\varepsilon-\varepsilon\sum\limits_{i=1}^{(n-1)/2}\left(C_{i-p-1}^{i}\right)\frac{w(i)}{w(0)+2\sum\limits_{h=1}^{\alpha(i+1)}w(h)}\\ -\varepsilon\frac{w(1)}{w(0)+2w(1)}-\varepsilon\sum\limits_{i=1}^{(n-3)/2}\left(C_{i-p-1}^{i}\right)\frac{w(i+1)}{w(0)+2\sum\limits_{h=1}^{\alpha(i+2)}w(h)}\\ \vdots\\ -\varepsilon\frac{w((n-3)/2)}{w(0)+2\sum\limits_{h=1}^{\alpha((n-1)/2)}w(h)}+\varepsilon p\frac{w((n-1)/2)}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)}w(h)}\\ -\varepsilon\frac{w((n-1)/2)}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)}w(h)}\\ 0\\ \vdots\\ 0 \end{bmatrix}.$$

Then calculate matrix norm $||\Delta E_v||_2$ and vector norm $||\Delta U_v||$, extract parameter ε and apply Lemma 3.1, parameter expressions in equation (29) are obtained. ■

Theorem 4.6 *If raw data disturbance occurs in the non-boundary nodes of the first half of the original sequence, that is $\widetilde{x}^{(0)}(k)=x^{(0)}(k)+\varepsilon$, then the difference between $\widetilde{\tau}_v$ and τ_v satisfies expression (27), but the values of parameters P_v and Q_v are determined by the following scenarios. Let parameters φ_1 and φ_2 be the same as those defined in Theorem 3.7, then P_v and Q_v are:*

(i). if sequence size n is even, then $k=2,3,...n/2$ and

$$P_v=\frac{1}{2}\sqrt{\left(\frac{w(|k-\varphi_1-1|)}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+1)}w(h)}\right)^2+\sum_{i=\varphi_1+1}^{n/2+\varphi_1-1}\left(\frac{w(|k-i-1|)}{w(0)+2\sum\limits_{h=1}^{\alpha(i+1)}w(h)}+\frac{w(|k-i|)}{w(0)+2\sum\limits_{h=1}^{\alpha(i)}w(h)}\right)^2+\left(\frac{w(|k-n/2-\varphi_1|)}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+\varphi_1)}w(h)}\right)^2},$$

$$Q_v=\sqrt{\sum_{i=1}^{\varphi_1}\left(\sum_{j=\varphi_1+1}^{n/2+\varphi_1}C_{j-p-2}^{j-1}\frac{w(|k-j|)}{w(0)+2\sum\limits_{h=1}^{\alpha(j)}w(h)}\right)^2+\sum_{i=\varphi_1+1}^{n/2+\varphi_1-1}\left(\sum_{j=i+1}^{n/2+\varphi_1}C_{j-p-2}^{j-1}\frac{w(|k-j|)}{w(0)+2\sum\limits_{h=1}^{\alpha(j)}w(h)}+\frac{w(|k-i|)}{w(0)+2\sum\limits_{h=1}^{\alpha(i)}w(h)}\right)^2+\left(\frac{w(|k-n/2-\varphi_1|)}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+\varphi_1)}w(h)}\right)^2}. \tag{30}$$

(ii). if sequence size n is odd, then $k = 2, 3, ...(n+1)/2$ and

$$\begin{aligned}
P_v =& \frac{1}{2}\sqrt{\left(\frac{w(|k-\varphi_1-1|)}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+1)} w(h)}\right)^2 + \sum_{i=\varphi_1+1}^{(n+1)/2+\varphi_2-1}\left(\frac{w(|k-i-1|)}{w(0)+2\sum\limits_{h=1}^{\alpha(i+1)} w(h)} + \frac{w(|k-i|)}{w(0)+2\sum\limits_{h=1}^{\alpha(i)} w(h)}\right)^2} \\
&\overline{+\left(\frac{w(|k-(n+1)/2-\varphi_2|)}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2+\varphi_2)} w(h)}\right)^2}, \\
Q_v =& \sqrt{\sum_{i=1}^{\varphi_1}\left(\sum_{j=\varphi_1+1}^{(n+1)/2+\varphi_2} C_{j-p-2}^{j-1}\frac{w(|k-j|)}{w(0)+2\sum\limits_{h=1}^{\alpha(j)} w(h)}\right)^2 + \left(\frac{w(|k-(n+1)/2-\varphi_2|)}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2+\varphi_2)} w(h)}\right)^2} \\
&\overline{+\sum_{i=\varphi_1+1}^{(n+1)/2+\varphi_2-1}\left(\sum_{j=i+1}^{(n+1)/2+\varphi_2} C_{j-p-2}^{j-1}\frac{w(|k-j|)}{w(0)+2\sum\limits_{h=1}^{\alpha(j)} w(h)} + \frac{w(|k-i|)}{w(0)+2\sum\limits_{h=1}^{\alpha(i)} w(h)}\right)^2}.
\end{aligned} \tag{31}$$

Proof The proof is similar to that of Theorem 3.7 and we omit the details here. ■

Theorem 4.7 *If raw data disturbance occurs in the second half of the original sequence, excluding the last point, let φ_3 and φ_4 be the same as those defined in Theorem 3.8, then the difference between $\widetilde{\tau}_v$ and τ_v satisfies expression (27), where parameters P_v and Q_v are determined by*

(i). if sequence size n is even, then $k = n/2+1, n/2+2, ...n-1$ and

$$\begin{aligned}
P_v =& \frac{1}{2}\sqrt{\left(\frac{w(|k+\varphi_4-n/2-1|)}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1-\varphi_4)} w(h)}\right)^2 + \sum_{i=n/2+1-\varphi_4}^{n-\varphi_4-1}\left(\frac{w(|k-i-1|)}{w(0)+2\sum\limits_{h=1}^{\alpha(i+1)} w(h)} + \frac{w(|k-i|)}{w(0)+2\sum\limits_{h=1}^{\alpha(i)} w(h)}\right)^2} \\
&\overline{\left(\frac{w(|k+\varphi_4-n|)}{w(0)+2\sum\limits_{h=1}^{\alpha(n-\varphi_4)} w(h)}\right)^2},
\end{aligned} \tag{32}$$

$$Q_v=\sqrt{\begin{array}{l}\displaystyle\sum_{i=1}^{n/2-\varphi_4}\left(\sum_{j=n/2+1-\varphi_4}^{n-\varphi_4}C_{j-p-2}^{j-1}\frac{w(|k-j|)}{w(0)+2\sum\limits_{h=1}^{\alpha(j)}w(h)}\right)^2+\left(\frac{w(|k+\varphi_4-n|)}{w(0)+2\sum\limits_{h=1}^{\alpha(n-\varphi_4)}w(h)}\right)^2\\ \displaystyle+\sum_{i=n/2+1-\varphi_4}^{n-\varphi_4-1}\left(\sum_{j=i+1}^{n-\varphi_4}C_{j-p-2}^{j-1}\frac{w(|k-j|)}{w(0)+2\sum\limits_{h=1}^{\alpha(j)}w(h)}+\frac{w(|k-i|)}{w(0)+2\sum\limits_{h=1}^{\alpha(i)}w(h)}\right)^2\end{array}}.$$

(ii). if sequence size n is odd, then $k=(n+1)/2+1,(n+1)/2+2,\ldots n-1$ and

$$\begin{aligned}P_v&=\frac{1}{2}\sqrt{\begin{array}{l}\displaystyle\left(\frac{w(|k+\varphi_3-(n+1)/2|)}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2-\varphi_3)}w(h)}\right)^2+\sum_{i=(n+1)/2-\varphi_3}^{n-\varphi_4-1}\left(\frac{w(|k-i-1|)}{w(0)+2\sum\limits_{h=1}^{\alpha(i+1)}w(h)}+\frac{w(|k-i|)}{w(0)+2\sum\limits_{h=1}^{\alpha(i)}w(h)}\right)^2\\ \displaystyle+\left(\frac{w(|k+\varphi_4-n|)}{w(0)+2\sum\limits_{h=1}^{\alpha(n-\varphi_4)}w(h)}\right)^2\end{array}},\\ Q_v&=\sqrt{\begin{array}{l}\displaystyle\sum_{i=1}^{(n-1)/2-\varphi_3}\left(\sum_{j=(n+1)/2-\varphi_3}^{n-\varphi_4}C_{j-p-2}^{j-1}\frac{w(|k-j|)}{w(0)+2\sum\limits_{h=1}^{\alpha(j)}w(h)}\right)^2+\left(\frac{w(|k+\varphi_4-n|)}{w(0)+2\sum\limits_{h=1}^{\alpha(n-\varphi_4)}w(h)}\right)^2\\ \displaystyle\sum_{i=(n+1)/2-\varphi_3}^{n-\varphi_4-1}\left(\sum_{j=i+1}^{n-\varphi_4}C_{j-p-2}^{j-1}\frac{w(|k-j|)}{w(0)+2\sum\limits_{h=1}^{\alpha(j)}w(h)}+\frac{w(|k-i|)}{w(0)+2\sum\limits_{h=1}^{\alpha(i)}w(h)}\right)^2\end{array}}.\end{aligned}\tag{33}$$

Proof The proof is similar to that of Theorem 3.8 and we omit the details here. ■

Theorem 4.8 *If disturbance occurs in the last data of the original sequence: $\widetilde{x}^{(0)}(n)=x^{(0)}(n)+\varepsilon$, then the difference between $\widetilde{\tau}_v$ and τ_v satisfies expression (27) with the following values of parameters P_v and Q_v:*

(i). if sequence size n is even, then

$$P_v = \frac{1}{2}\sqrt{\left(\frac{w(n/2-1)}{w(0)+2\sum_{h=1}^{\alpha(n/2+1)} w(h)}\right)^2 + \sum_{i=n/2+1}^{n-1}\left(\frac{w(n-i-1)}{w(0)+2\sum\limits_{h=1}^{\alpha(i+1)} w(h)} + \frac{w(n-i)}{w(0)+2\sum\limits_{h=1}^{\alpha(i)} w(h)}\right)^2},$$

$$Q_v = \sqrt{\sum_{i=1}^{n/2}\left(\sum_{j=n/2+1}^{n} C_{j-p-2}^{j-1}\frac{w(n-j)}{w(0)+2\sum\limits_{h=1}^{\alpha(j)} w(h)}\right)^2 + \sum_{i=n/2+1}^{n-1}\left(\frac{w(n-i)}{w(0)+2\sum\limits_{h=1}^{\alpha(i)} w(h)} + \sum_{j=i+1}^{n} C_{j-p-2}^{j-1}\frac{w(n-j)}{w(0)+2\sum\limits_{h=1}^{\alpha(j)} w(h)}\right)^2}. \tag{34}$$

(ii). if sequence size n is odd, then

$$P_v = \frac{1}{2}\sqrt{\left(\frac{w((n-1)/2)}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)} w(h)}\right)^2 + \sum_{i=(n+1)/2}^{n-1}\left(\frac{w(n-i-1)}{w(0)+2\sum\limits_{h=1}^{\alpha(i+1)} w(h)} + \frac{w(n-i)}{w(0)+2\sum\limits_{h=1}^{\alpha(i)} w(h)}\right)^2},$$

$$Q_v = \sqrt{\sum_{i=1}^{(n-1)/2}\left(\sum_{j=(n+1)/2}^{n} C_{j-p-2}^{j-1}\frac{w(n-j)}{w(0)+2\sum_{h=1}^{\alpha(j)} w(h)}\right)^2 + \sum_{i=(n+1)/2}^{n-1}\left(\frac{w(n-i)}{w(0)+2\sum\limits_{h=1}^{\alpha(i)} w(h)} + \sum_{j=i+1}^{n} C_{j-p-2}^{j-1}\frac{w(n-j)}{w(0)+2\sum\limits_{h=1}^{\alpha(j)} w(h)}\right)^2}. \tag{35}$$

Proof The proof is similar to that of Theorem 3.9 and we omit the details here. ■

5 Numerical performance of operator D_v in improving model stability

5.1 Numerical study on model $DGM^p(1,1)$

To evaluate the effect of the fractional bidirectional weakening buffer operator D_v in improving stability of model $DGM^p(1,1)$, we consider a real case presented in [18]. This example predicts the annual cargo turnover in Jiangsu, one big coastal province in China. The original data are shown in Table 1 and the forecasting values of four compared models are presented in Table 2.

Table 1 The original data of the annual cargo turnover used in [18]. (unit: 100 million ton-km)

	Modeling period						Forecasting period
	2003	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**
Freight Ton-Kilometers	1817.44	2398.13	3068.3	3644.14	4098.42	4707.5	5154.46

Table 2 The forecasting results of four different models.

	$DGM^{1/2}(1,1)^*$	D_v+$DGM^{1/2}(1,1)$	$DGM^{2/3}(1,1)^*$	D_v+$DGM^{2/3}(1,1)$
Freight Ton-Kilometers(2009)	5236.45	5238.69	5303.33	5305.58
MAPE(%)	1.59	1.63	2.89	2.93

$*$ values obtained from [18].

Table 2 reveals that forecasting results of the $DGM^p(1,1)$ model with data preprocessing based on $D_v(v = 0.02)$ are very close to those without D_v, which means the validity of the predicted values is kept.

Then data noise interferences with different noise amplitudes (range from -10% to 10% of the original value) are studied. The effectiveness of operator D_v on prediction model is evaluated via the variation of predicted values (VP) with and without noise interference. Let $y_0(i)(i = 1, \dots n)$ be the predicted value based on the original modeling data without noise interference and $y_1(i)(i = 1, \dots n)$ be the predicted value with data noise, then we define $VP = (\sum_{i=1}^{n} |y_1(i) - y_0(i)|)/n$ and ΔVP=$VP(DGM^p(1,1))$−$VP(D_v$+$DGM^p(1,1))$. If ΔVP falls into the positive domain, it means D_v improves the stability of model $DGM^p(1,1)$. Results are shown in Table 3.

Table 3 The ΔVP values of $DGM^p(1,1)$ in the first noise scenario.

Model parameter	**Noise parameter**	**Noise amplitude** -10%	-8%	-6%	-4%	-2%	2%	4%	6%	8%	10%
$p = 0.5$	**2003**	2.6628	2.1243	1.5884	1.0552	0.5247	0.5285	1.0513	1.5715	2.0892	2.6044
	2004	1.7216	1.5510	1.2953	0.9532	0.5230	0.6081	1.3126	2.1123	3.0087	4.0037
	2005	0.7736	0.9582	0.9699	0.8125	0.4891	0.6420	1.4430	2.3964	3.4982	4.7444
	2006	0.2291	0.6812	0.8729	0.8164	0.5230	0.7329	1.6756	2.8156	4.1435	5.6502
	2007	12.6333	9.5106	6.7037	4.1937	1.9632	1.7191	3.1968	4.4470	5.4795	6.3032
	2008	-3.4280	-2.7186	-2.0199	-1.3334	-0.6607	-0.6377	-1.2610	-1.8653	-2.4499	-3.0140
$p = 2/3$	**2003**	2.8174	2.2531	1.6890	1.1250	0.5612	0.5660	1.1295	1.6928	2.2561	2.8194
	2004	2.7897	2.3443	1.8434	1.2866	0.6732	0.7262	1.5134	2.3598	3.2661	4.2327
	2005	1.5322	1.4354	1.2316	0.9234	0.5129	0.6054	1.3085	2.1042	2.9900	3.9635
	2006	0.6858	0.8631	0.8772	0.7345	0.4410	0.5755	1.2875	2.1284	3.0930	4.1764
	2007	12.5607	9.6861	6.9995	4.4937	2.1617	2.0047	3.8503	5.5440	7.0902	8.4931
	2008	-4.5834	-3.6746	-2.7613	-1.8443	-0.9244	-0.9208	-1.8446	-2.7685	-3.6919	-4.6142

It shows that operator D_v improves the stability of prediction performance in most noise samples, except for noises added on the last modeling point (year 2008). Variations of predicted values increase with increasing noise amplitudes, but data preprocessing based on D_v effectively reduce the noise effect on prediction results. Though D_v fails when noise occurs in the last modeling point, data interference occurs in other points are well controlled.

5.2 Numerical study on model $NIGM(p,1)$

We now consider the effect of operator $D_v(v = 0.06)$ in improving the stability of model $NIGM(p,1)$ in forecasting the annual electricity consumption in Russia[19]. The forecasting results of four compared models without data noise interference are shown in Table 4. It is clear that operator D_v improves the short-term (2004-2005) forecasting of $NIGM(p,1)$. Though its long-term (2006-2007) prediction performance is worse than those from $NIGM(p,1)$, its prediction accuracy is still acceptable and higher than models $DGM(1,1)$ and $GM(0.98,1)$.

Table 4 The forecasting results of $NIGM(p,1)$ models with and without operator D_v.

	Modeling period				Forecasting period			
	2000	**2001**	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**
Actual value	52333	53151	53168	54372	55516	55898	58600	60281
$DGM(1,1)^*$	52333	52953	53561	54176	54798	55428	56065	56709
$GM(0.98,1)^*$	52333	53704	54858	55849	56704	57445	58087	58645
$NIGM(0.997,1)^*$	52333	52627	53051	53666	54559	55855	57736	60464
$D_v + NIGM(0.997,1)$	52333	52701	53193	53854	54742	55934	57534	59684

$*$ values obtained from [19].

Applying the indicator VP defined in Section 5.1, the difference between $NIGM(p,1)$ with and without D_v can be expressed by $\Delta VP{=}VP(NIGM(p,1)){-}VP(D_v{+}NIGM(p,1))$. Table 5 shows ΔVP results in different noise amplitude scenarios. We can see that, when data disturbance occurs in early modeling period (2000 or 2001 in this case), operator D_v improves the stability of prediction results in all thirty noise amplitude conditions. When noise occurs in the later modeling time point 2002 (or 2003), prediction without D_v outperforms that with D_v in only four (or five) out of thirty noise perturbations. These confirm the effectiveness of operator D_v in improving the stability of prediction model $NIGM(p,1)$.

Numerical cases in this section demonstrate that the fractional bidirectional weakening buffer operator D_v reduces the one-sided reaction to sample disturbances, and on the other hand it can effectively improve the stability of time series prediction models by taking into account both the old and the new information.

Table 5 The ΔVP values of $NIGM(p, 1)$ in the first noise scenario.

Noise position	**Noise amplitude(%)**									
	-0.03	**-0.028**	**-0.026**	**-0.024**	**-0.022**	**-0.020**	**-0.018**	**-0.016**	**-0.014**	**-0.012**
2000	26.90	36.95	46.95	56.91	66.82	76.69	86.51	96.28	106.00	115.66
2001	162.80	163.51	164.21	164.91	165.59	166.27	166.94	167.60	168.25	168.89
2002	143.20	144.05	145.33	148.15	150.84	153.40	155.83	158.14	160.33	162.40
2003	-85.52	-68.22	-50.94	-33.66	-16.40	0.85	18.09	35.31	52.52	69.71
	-0.010	**-0.008**	**-0.006**	**-0.004**	**-0.002**	**0.002**	**0.004**	**0.006**	**0.008**	**0.010**
2000	125.28	134.84	144.35	153.80	163.20	170.73	168.87	166.94	164.95	162.89
2001	169.52	170.14	170.75	171.35	171.95	162.74	152.94	143.13	133.30	123.47
2002	164.36	166.21	167.94	169.58	171.10	158.24	143.88	129.45	114.96	100.40
2003	86.89	104.05	121.19	138.32	155.44	172.27	171.98	171.66	171.31	170.94
	0.012	**0.014**	**0.016**	**0.018**	**0.020**	**0.022**	**0.024**	**0.026**	**0.028**	**0.030**
2000	160.77	158.57	156.31	153.98	151.58	149.10	146.55	143.93	141.23	138.45
2001	113.62	103.76	93.89	84.01	74.12	64.21	54.29	44.37	34.42	24.47
2002	85.78	71.10	56.37	41.58	26.74	11.86	-3.07	-18.04	-33.06	-43.95
2003	170.53	170.10	169.64	169.15	168.63	168.08	167.50	166.89	166.24	165.57

6 Conclusions

The performance of sequence operators in the stability of operator-based time series models has a direct impact on the validity of model findings. Without a specific requirement for a particular part of information in a series object, all data elements in this series should be treated equally. However, some widely used weakening buffer operators are too subjectively biased in dealing with samples and sample disturbances. In contrast, the fractional bidirectional weakening buffer operator D_v presents a more objective approach in data processing. Both theoretical investigations and numerical experiments based on real cases show a favorable effectiveness of the operator D_v in improving the stability of time series models. Other features of the operator and their effects in time series models worth further studies.

Acknowledgements The authors would like to thank the editors and reviewers for their valuable comments and suggestions on this paper.

References

[1] Yu L, Wang S, Lai K K, et al., Time series forecasting with multiple candidate models: selecting or combining?, *Journal of Systems Science and Complexity*, 2005, **18**: 1–18.

[2] Deng G, Chen G, and Qian L, On the global asymptotic stability and oscillation of solutions in a stochastic business cycle model, *Journal of Difference Equations and Applications*, 2016, **22**: 1609–1620.

[3] Braverman E and Rodkina A, On convergence of solutions to difference equations with additive perturbations, *Journal of Difference Equations and Applications*, 2016, **22**: 878–903.

[4] Zhu M, Wang D, and Guo M, Existence and stability of stationary solutions of nonlinear difference equations under random perturbations, *Journal of Difference Equations and Applications*, 2011, **17**: 587–602.

[5] Yu L, Wang S, and Lai K K, Forecasting Chinas foreign trade volume with a kernel-based hybrid econometric-AI ensemble learning approach, *Journal of Systems Science and Complexity*, 2008, **21**: 1–19.

[6] Chandra R and Zhang M, Cooperative coevolution of Elman recurrent neural networks for chaotic time series prediction, *Neurocomputing*, 2012, **86**: 116–123.

[7] Yager R R, Exponential smoothing with credibility weighted observations, *Information Sciences*, 2013, **252**: 96–105.

[8] Rasmussen R, On time series data and optimal parameters, *Omega*, 2004, **32**: 111–120.

[9] Taylor J W and Snyder R D, Forecasting intraday time series with multiple seasonal cycles using parsimonious seasonal exponential smoothing, *Omega*, 2012, **40**: 748–757.

[10] Liu S F and Forrest J Y L, *Grey Systems: Theory and Applications*, Springer-Verlag, Berlin, Heidelberg, 2010.

[11] Hu Y C, Grey prediction with residual modification using functional-link net and its application to energy demand forecasting, *Kybernetes*, 2017, **46**: 349–363.

[12] Lin Y, Chen M, and Liu S, Theory of grey systems: capturing uncertainties of grey information, *Kybernetes*, 2004, **33**: 196–218.

[13] Liu S F, Yang Y J, and Forrest J, *Grey Data Analysis: Methods, Models and Applications*, Springer, Singapore, 2016.

[14] Dang Y G, Liu S F, Liu B, et al., Study on the buffer weakening operator, *Chinese Journal of Management Science*, 2004, **12**: 108–111.

[15] Xie N M and Liu S F, A new applicative weakening buffer operator, *Chinese Journal of Management Science*, 2003, **11**: 46–48.

[16] Mao S, Gao M, Xiao X, et al., A novel fractional grey system model and its application, *Applied Mathematical Modelling*, 2016, **40**: 5063–5076.

[17] Wu L, Liu S, Yang Y, et al., Multi-variable weakening buffer operator and its application, *Information Sciences*, 2016, **339**: 98–107.

[18] Wu L F, Liu S F, and Yao L G, Discrete grey model based on fractional order accumulate, *Systems Engineering-Theory & Practice*, 2014, **34**: 1822–1827.

[19] Wu L F, Liu S F and Yao L G, Grey model with Caputo fractional order derivative, *System Engineering-Theory & Practice*, 2015, **35**: 1311–1316.

[20] Li N, Wang J, Wu L, et al., Predicting monthly natural gas production in China using a novel grey seasonal model with particle swarm optimization, *Energy*, 2021, **215**: 119118.

[21] Tang X, Xie N, and Hu A, Forecasting annual foreign tourist arrivals to China by incorporating firefly algorithm into fractional non-homogenous discrete Grey model, *Kybernetes*, 2022, **51**: 676–693.

[22] Yuan Y, Zhao H, Yuan X, et al., Application of fractional order-based grey power model in water consumption prediction, *Environmental Earth Sciences*, 2019, **78**: 1–8.

[23] Li C, Yang Y, and Liu S. Comparative analysis of properties of weakening buffer operators in time series prediction models, *Communications in Nonlinear Science and Numerical Simulation*, 2019, **68**: 257–285.

[24] Li C, Yang Y, and Liu S, A new method to mitigate data fluctuations for time series prediction, *Applied Mathematical Modelling*, 2019, **65**: 390–407.

[25] Xiao X, Guo H, Mao S. The modeling mechanism, extension and optimization of grey GM (1, 1) model, *Applied Mathematical Modelling*, 2014, **38**: 1896–1910.

[26] Stewart G W and Sun J, *Matrix perturbation theory*, Academic press, 1990.

APPENDIX

In this appendix, we provide the proof of error bound theorems in $DGM^p(1,1)$ model.

A.1 Proof of Theorem 3.7

proof (i). Sequence size n is even and $k = 2, 3, ..., n/2$. According to the proposed operator D_v, sequences with and without disturbance applied to $DGM^p(1,1)$ satisfy:

$$\widetilde{x}_v(i) = \begin{cases} x_v(i), & \text{when} \quad 1 \leq i \leq \varphi_1 \quad \text{or} \quad n/2 + \varphi_1 < i \leq n \\ x_v(i) + \frac{\varepsilon w(|i-k|)}{\left(w(0)+2\sum_{j=1}^{\alpha(i)} w(j)\right)}, & \text{when} \quad \varphi_1 < i \leq n/2 + \varphi_1 \end{cases} \tag{A1}$$

Then we can obtain:

$$\Delta B_v = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+1)} w(h)} \\ 0 & C_p^1 \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+1)} w(h)} + \frac{w(|k-\varphi_1-2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+2)} w(h)} \\ \vdots & \vdots \\ 0 & C_{p+n/2-2}^{n/2-1} \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+1)} w(h)} + C_{p+n/2-3}^{n/2-2} \frac{w(|k-\varphi_1-2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+2)} w(h)} + \cdots \frac{w(|k-\varphi_1-n/2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+\varphi_1)} w(h)} \\ 0 & C_{p+n/2-1}^{n/2} \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+1)} w(h)} + C_{p+n/2-2}^{n/2-1} \frac{w(|k-\varphi_1-2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+2)} w(h)} + \cdots C_p^1 \frac{w(|k-\varphi_1-n/2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+\varphi_1)} w(h)} \\ \vdots & \vdots \\ 0 & C_{p+n-\varphi_1-3}^{n-\varphi_1-2} \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+1)} w(h)} + C_{p+n-\varphi_1-4}^{n-\varphi_1-3} \frac{w(|k-\varphi_1-2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+2)} w(h)} + \cdots C_{p+n/2-\varphi_1-2}^{n/2-\varphi_1-1} \frac{w(|k-\varphi_1-n/2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+\varphi_1)} w(h)} \end{bmatrix},$$

$$
\Delta Y_v = \begin{bmatrix}
0 \\
\vdots \\
0 \\
\frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+1)} w(h)} \\
C_p^1 \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+1)} w(h)} + \frac{w(|k-\varphi_1-2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+2)} w(h)} \\
\vdots \\
C_{p+n/2-2}^{n/2-1} \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+1)} w(h)} + C_{p+n/2-3}^{n/2-2} \frac{w(|k-\varphi_1-2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+2)} w(h)} + \cdots \frac{w(|k-\varphi_1-n/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(n/2+\varphi_1)} w(h)} \\
C_{p+n/2-1}^{n/2} \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+1)} w(h)} + C_{p+n/2-2}^{n/2-1} \frac{w(|k-\varphi_1-2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+2)} w(h)} + \cdots C_p^1 \frac{w(|k-\varphi_1-n/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(n/2+\varphi_1)} w(h)} \\
\vdots \\
C_{p+n-\varphi_1-2}^{n-\varphi_1-1} \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+1)} w(h)} + C_{p+n-\varphi_1-3}^{n-\varphi_1-2} \frac{w(|k-\varphi_1-2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+2)} w(h)} + \cdots C_{p+n/2-\varphi_1-1}^{n/2-\varphi_1} \frac{w(|k-\varphi_1-n/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(n/2+\varphi_1)} w(h)}
\end{bmatrix}.
$$

Then calculate matrix norm $||\Delta B_v||_2$and vector norm $||\Delta Y_v||$, extract parameter ε and apply Lemma 3.1, parameter expressions in equation (15) are obtained.

(ii). Sequence size n is odd and $k = 2, 3, ..., (n+1)/2$. According to the fractional bidirectional weakening buffer operator D_v, sequences with and without disturbance applied to $DGM^p(1,1)$ satisfy:

$$
\widetilde{x}_v(i) = \begin{cases} x_v(i), & \text{when} \quad 1 \le i \le \varphi_1 \quad \text{or} \quad (n+1)/2 + \varphi_2 < i \le n \\ x_v(i) + \frac{\varepsilon w(|i-k|)}{\left(w(0)+2\sum_{j=1}^{\alpha(i)} w(j)\right)}, & \text{when} \quad \varphi_1 < i \le (n+1)/2 + \varphi_2 \end{cases} \tag{A2}
$$

Then we can obtain:

$$
\Delta B_v = \begin{bmatrix}
0 & 0 \\
\vdots & \vdots \\
0 & 0 \\
0 & \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+1)} w(h)} \\
0 & C_p^1 \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+1)} w(h)} + \frac{w(|k-\varphi_1-2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+2)} w(h)} \\
\vdots & \vdots \\
0 & C_{p+(n+1)/2+\varphi_2-\varphi_1-2}^{(n+1)/2+\varphi_2-\varphi_1-1} \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+1)} w(h)} + C_{p+(n+1)/2+\varphi_2-\varphi_1-3}^{(n+1)/2+\varphi_2-\varphi_1-2} \frac{w(|k-\varphi_1-2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+2)} w(h)} + \cdots \frac{w(|k-\varphi_2-(n+1)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+1)/2+\varphi_2)} w(h)} \\
0 & C_{p+(n+)/2+\varphi_2-\varphi_1-1}^{(n+1)/2+\varphi_2-\varphi_1} \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+1)} w(h)} + C_{p+(n+1)/2+\varphi_2-\varphi_1-2}^{(n+1)/2+\varphi_2-\varphi_1-1} \frac{w(|k-\varphi_1-2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+2)} w(h)} + \cdots C_p^1 \frac{w(|k-\varphi_2-(n+1)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+1)/2+\varphi_2)} w(h)} \\
\vdots & \vdots \\
0 & C_{p+n-\varphi_1-3}^{n-\varphi_1-2} \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+1)} w(h)} + C_{p+n-\varphi_1-4}^{n-\varphi_1-3} \frac{w(|k-\varphi_1-2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(\varphi_1+2)} w(h)} + \cdots C_{p+(n-1)/2-\varphi_2-2}^{(n-1)/2-\varphi_2-1} \frac{w(|k-\varphi_2-(n+1)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+1)/2+\varphi_2)} w(h)}
\end{bmatrix},
$$

$$\Delta Y_v = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+1)} w(h)} \\ C_p^1 \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+1)} w(h)} + \frac{w(|k-\varphi_1-2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+2)} w(h)} \\ \vdots \\ C_{p+(n+1)/2+\varphi_2-\varphi_1-2}^{(n+1)/2+\varphi_2-\varphi_1-1} \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+1)} w(h)} + C_{p+(n+1)/2+\varphi_2-\varphi_1-3}^{(n+1)/2+\varphi_2-\varphi_1-2} \frac{w(|k-\varphi_1-2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+2)} w(h)} + \cdots \frac{w(|k-\varphi_2-(n+1)/2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2+\varphi_2)} w(h)} \\ C_{p+(n+1)/2+\varphi_2-\varphi_1-1}^{(n+1)/2+\varphi_2-\varphi_1} \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+1)} w(h)} + C_{p+(n+1)/2+\varphi_2-\varphi_1-2}^{(n+1)/2+\varphi_2-\varphi_1-1} \frac{w(|k-\varphi_1-2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+2)} w(h)} + \cdots C_p^1 \frac{w(|k-\varphi_2-(n+1)/2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2+\varphi_2)} w(h)} \\ \vdots \\ C_{p+n-\varphi_1-2}^{n-\varphi_1-1} \frac{w(|k-\varphi_1-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+1)} w(h)} + C_{p+n-\varphi_1-3}^{n-\varphi_1-2} \frac{w(|k-\varphi_1-2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(\varphi_1+2)} w(h)} + \cdots C_{p+(n-1)/2-\varphi_2-1}^{(n-1)/2-\varphi_2} \frac{w(|k-\varphi_2-(n+1)/2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2+\varphi_2)} w(h)} \end{bmatrix}.$$

Then calculate matrix norm $||\Delta B_v||_2$ and vector norm $||\Delta Y_v||$, extract parameter ε and apply Lemma 3.1, parameter expressions in equation (16) are obtained. The proof of Theorem 3.7 is now complete. ■

A.2 Proof of Theorem 3.8

proof (i). Sequence size n is even and $k = n/2+1, n/2+2, ..., n-1$. According to the fractional bidirectional weakening buffer operator D_v, sequences with and without disturbance applied to $DGM^p(1,1)$ satisfy:

$$\widetilde{x}_v(i) = \begin{cases} x_v(i), & \text{when} \quad 1 \le i < n/2+1-\varphi_4 \quad \text{or} \quad n+1-\varphi_4 \le i \le n \\ x_v(i) + \frac{\varepsilon w(|i-k|)}{\left(w(0)+2\sum_{j=1}^{\alpha(i)} w(j)\right)}, & \text{when} \quad n/2+1-\varphi_4 \le i < n+1-\varphi_4 \end{cases} \tag{A3}$$

Then we can obtain:

$$\Delta B_v = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & \frac{w(|k+\varphi_4-n/2-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1-\varphi_4)} w(h)} \\ 0 & C_p^1 \frac{w(|k+\varphi_4-n/2-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1-\varphi_4)} w(h)} + \frac{w(|k+\varphi_4-n/2-2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+2-\varphi_4)} w(h)} \\ \vdots & \vdots \\ 0 & C_{p+n/2-2}^{n/2-1} \frac{w(|k+\varphi_4-n/2-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1-\varphi_4)} w(h)} + C_{p+n/2-3}^{n/2-2} \frac{w(|k+\varphi_4-n/2-2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+2-\varphi_4)} w(h)} + \cdots \frac{w(|k+\varphi_4-n|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n-\varphi_4)} w(h)} \\ 0 & C_{p+n/2-1}^{n/2} \frac{w(|k+\varphi_4-n/2-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1-\varphi_4)} w(h)} + C_{p+n/2-2}^{n/2-1} \frac{w(|k+\varphi_4-n/2-2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+2-\varphi_4)} w(h)} + \cdots C_p^1 \frac{w(|k+\varphi_4-n|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n-\varphi_4)} w(h)} \\ \vdots & \vdots \\ 0 & C_{p+n/2+\varphi_4-3}^{n/2+\varphi_4-2} \frac{w(|k+\varphi_4-n/2-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1-\varphi_4)} w(h)} + C_{p+n/2+\varphi_4-4}^{n/2+\varphi_4-3} \frac{w(|k+\varphi_4-n/2-2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+2-\varphi_4)} w(h)} + \cdots C_{p+\varphi_4-2}^{\varphi_4-1} \frac{w(|k+\varphi_4-n|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n-\varphi_4)} w(h)} \end{bmatrix},$$

$$\Delta Y_v = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{w(|k+\varphi_4-n/2-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1-\varphi_4)} w(h)} \\ C_p^1 \frac{w(|k+\varphi_4-n/2-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1-\varphi_4)} w(h)} + \frac{w(|k+\varphi_4-n/2-2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+2-\varphi_4)} w(h)} \\ \vdots \\ C_{p+n/2-2}^{n/2-1} \frac{w(|k+\varphi_4-n/2-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1-\varphi_4)} w(h)} + C_{p+n/2-3}^{n/2-2} \frac{w(|k+\varphi_4-n/2-2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+2-\varphi_4)} w(h)} + \cdots \frac{w(|k+\varphi_4-n|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n-\varphi_4)} w(h)} \\ C_{p+n/2-1}^{n/2} \frac{w(|k+\varphi_4-n/2-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1-\varphi_4)} w(h)} + C_{p+n/2-2}^{n/2-1} \frac{w(|k+\varphi_4-n/2-2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+2-\varphi_4)} w(h)} + \cdots C_p^1 \frac{w(|k+\varphi_4-n|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n-\varphi_4)} w(h)} \\ \vdots \\ C_{p+n/2+\varphi_4-2}^{n/2+\varphi_4-1} \frac{w(|k+\varphi_4-n/2-1|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1-\varphi_4)} w(h)} + C_{p+n/2+\varphi_4-3}^{n/2+\varphi_4-2} \frac{w(|k+\varphi_4-n/2-2|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+2-\varphi_4)} w(h)} + \cdots C_{p+\varphi_4-1}^{\varphi_4} \frac{w(|k+\varphi_4-n|)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n-\varphi_4)} w(h)} \end{bmatrix}.$$

Then calculate matrix norm $||\Delta B_v||_2$ and vector norm $||\Delta Y_v||$, extract parameter ε and apply Lemma 3.1, parameter expressions in equation (17) are obtained.

(ii). Sequence size n is odd and $k = (n+1)/2+1, (n+1)/2+2, ..., n-1$. Based on operator D_v, sequences with and without disturbance applied to $DGM^p(1,1)$ satisfy:

$$\widetilde{x}_v(i) = \begin{cases} x_v(i), & \text{when} \quad 1 \le i < (n+1)/2 - \varphi_3 \quad \text{or} \quad n+1-\varphi_4 \le i \le n \\ x_v(i) + \frac{\varepsilon w(|i-k|)}{\left(w(0)+2\sum_{j=1}^{\alpha(i)} w(j)\right)}, & \text{when} \quad (n+1)/2 - \varphi_3 \le i < n+1-\varphi_4 \end{cases} \tag{A4}$$

Then we can obtain:

$$\Delta B_v = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & \frac{w(|k+\varphi_3-(n+1)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+1)/2-\varphi_3)} w(h)} \\ 0 & C_p^1 \frac{w(|k+\varphi_3-(n+1)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+1)/2-\varphi_3)} w(h)} + \frac{w(|k+\varphi_3-(n+3)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+3)/2-\varphi_3)} w(h)} \\ \vdots & \vdots \\ 0 & C_{p+(n-3)/2+\varphi_3-\varphi_4}^{(n-1)/2+\varphi_3-\varphi_4} \frac{w(|k+\varphi_3-(n+1)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+1)/2-\varphi_3)} w(h)} + C_{p+(n-5)/2+\varphi_3-\varphi_4}^{(n-3)/2+\varphi_3-\varphi_4} \frac{w(|k+\varphi_3-(n+3)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+3)/2-\varphi_3)} w(h)} + \cdots \frac{w(|k+\varphi_4-n|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(n-\varphi_4)} w(h)} \\ 0 & C_{p+(n-1)/2+\varphi_3-\varphi_4}^{(n+1)/2+\varphi_3-\varphi_4} \frac{w(|k+\varphi_3-(n+1)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+1)/2-\varphi_3)} w(h)} + C_{p+(n-3)/2+\varphi_3-\varphi_4}^{(n-1)/2+\varphi_3-\varphi_4} \frac{w(|k+\varphi_3-(n+3)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+3)/2-\varphi_3)} w(h)} + \cdots C_p^1 \frac{w(|k+\varphi_4-n|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(n-\varphi_4)} w(h)} \\ \vdots & \vdots \\ 0 & C_{p+(n-5)/2+\varphi_3}^{(n-3)/2+\varphi_3} \frac{w(|k+\varphi_3-(n+1)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+1)/2-\varphi_3)} w(h)} + C_{p+(n-7)/2+\varphi_3}^{(n-5)/2+\varphi_3} \frac{w(|k+\varphi_3-(n+3)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+3)/2-\varphi_3)} w(h)} + \cdots C_{p++\varphi_4-2}^{\varphi_4-1} \frac{w(|k+\varphi_4-n|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(n-\varphi_4)} w(h)} \end{bmatrix},$$

$$\Delta Y_v = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{w(|k+\varphi_3-(n+1)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+1)/2-\varphi_3)} w(h)} \\ C_p^1 \frac{w(|k+\varphi_3-(n+1)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+1)/2-\varphi_3)} w(h)} + \frac{w(|k+\varphi_3-(n+3)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+3)/2-\varphi_3)} w(h)} \\ \vdots \\ C_{p+(n-3)/2+\varphi_3-\varphi_4}^{(n-1)/2+\varphi_3-\varphi_4} \frac{w(|k+\varphi_3-(n+1)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+1)/2-\varphi_3)} w(h)} + C_{p+(n-5)/2+\varphi_3-\varphi_4}^{(n-3)/2+\varphi_3-\varphi_4} \frac{w(|k+\varphi_3-(n+3)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+3)/2-\varphi_3)} w(h)} + \cdots \frac{w(|k+\varphi_4-n|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(n-\varphi_4)} w(h)} \\ C_{p+(n-1)/2+\varphi_3-\varphi_4}^{(n+1)/2+\varphi_3-\varphi_4} \frac{w(|k+\varphi_3-(n+1)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+1)/2-\varphi_3)} w(h)} + C_{p+(n-3)/2+\varphi_3-\varphi_4}^{(n-1)/2+\varphi_3-\varphi_4} \frac{w(|k+\varphi_3-(n+3)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+3)/2-\varphi_3)} w(h)} + \cdots C_p^1 \frac{w(|k+\varphi_4-n|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(n-\varphi_4)} w(h)} \\ \vdots \\ C_{p+(n-3)/2+\varphi_3}^{(n-1)/2+\varphi_3} \frac{w(|k+\varphi_3-(n+1)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+1)/2-\varphi_3)} w(h)} + C_{p+(n-5)/2+\varphi_3}^{(n-3)/2+\varphi_3} \frac{w(|k+\varphi_3-(n+3)/2|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha((n+3)/2-\varphi_3)} w(h)} + \cdots C_{p+\varphi_4-1}^{\varphi_4} \frac{w(|k+\varphi_4-n|)\varepsilon}{w(0)+2\sum_{h=1}^{\alpha(n-\varphi_4)} w(h)} \end{bmatrix}.$$

Then calculate matrix norm $||\Delta B_v||_2$ and vector norm $||\Delta Y_v||$, extract parameter ε and apply Lemma 3.1, parameter expressions in equation (18) are obtained.

The proof of Theorem 3.8 is now complete. ■

A.3 Proof of Theorem 3.9

proof (i). When sequence size n is even, according to the logic of $DGM^p(1,1)$, we have :

$$\Delta B_v = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & \dfrac{w(n/2-1)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1)} w(h)} \\ 0 & C_p^1 \dfrac{w(n/2-1)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1)} w(h)} + \dfrac{w(n/2-2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+2)} w(h)} \\ \vdots & \vdots \\ 0 & C_{p+n/2-3}^{n/2-2} \dfrac{w(n/2-1)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1)} w(h)} + C_{p+n/2-4}^{n/2-3} \dfrac{w(n/2-2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+2)} w(h)} + \cdots \dfrac{w(1)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n-1)} w(h)} \end{bmatrix},$$

$$\Delta Y_v = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \dfrac{w(n/2-1)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1)} w(h)} \\ C_p^1 \dfrac{w(n/2-1)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1)} w(h)} + \dfrac{w(n/2-2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+2)} w(h)} \\ \vdots \\ C_{p+n/2-2}^{n/2-1} \dfrac{w(n/2-1)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+1)} w(h)} + C_{p+n/2-3}^{n/2-2} \dfrac{w(n/2-2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n/2+2)} w(h)} + \cdots \dfrac{w(0)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n)} w(h)} \end{bmatrix}.$$

Then calculate matrix norm $||\Delta B_v||_2$ and vector norm $||\Delta Y_v||$, extract parameter ε and apply Lemma 3.1, parameter expressions in equation (19) are obtained.

(ii). When sequence size n is odd, we can obtain:

$$\Delta B_v = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & \dfrac{w((n-1)/2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)} w(h)} \\ 0 & C_p^1 \dfrac{w((n-1)/2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)} w(h)} + \dfrac{w((n-3)/2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+3)/2)} w(h)} \\ \vdots & \vdots \\ 0 & C_{p+(n-5)/2}^{(n-3)/2} \dfrac{w((n-1)/2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)} w(h)} + C_{p+(n-7)/2}^{(n-5)/2} \dfrac{w((n-3)/2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+3)/2)} w(h)} + \cdots \dfrac{w(1)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n-1)} w(h)} \end{bmatrix},$$

$$\Delta Y_v = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{w((n-1)/2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)} w(h)} \\ C_p^1 \frac{w((n-1)/2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)} w(h)} + \frac{w((n-3)/2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+3)/2)} w(h)} \\ \vdots \\ C_{p+(n-3)/2}^{(n-1)/2} \frac{w((n-1)/2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+1)/2)} w(h)} + C_{p+(n-5)/2}^{(n-3)/2} \frac{w((n-3)/2)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha((n+3)/2)} w(h)} + \cdots \frac{w(0)\varepsilon}{w(0)+2\sum\limits_{h=1}^{\alpha(n)} w(h)} \end{bmatrix}.$$

Then calculate matrix norm $||\Delta B_v||_2$ and vector norm $||\Delta Y_v||$, extract parameter ε and apply Lemma 3.1, parameter expressions in equation (20) are obtained.

The proof of Theorem 3.9 is now complete. ■