Output Feedback Stabilization for Dynamic MIMO Semi-linear Stochastic Systems with Output Randomness Attenuation

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***Abstract*—In this paper, the problem of randomness attenuation is investigated for a class of MIMO semi-linear stochastic systems. To achieve this control objective, a m-block backstepping controller is designed to stabilize the closed-loop systems in probability sense. In addition, the output randomness attenuation can be achieved by optimising the design parameters using minimum entropy criterion. The effectiveness of this presented control algorithm can be verified by a given numerical example. In summary, the main contributions of this paper are characterized as follows: (1) an output feedback design method is adapted to stabilise the dynamic multi-variable semi-linear stochastic systems by block backstepping; (2) randomness of the system output is attenuated by searching the optimal design parameter based on the entropy criterion; (3) a framework of performance enhancement for stochastic systems is developed.**

I. Introduction

Backstepping design method [1] has been presented as an effective approach for the controller design of SISO deterministic nonlinear control systems. For MIMO deterministic non-linear systems, [2][3] presented various controllers based on backstepping design. Recently, block backstepping design is investigated by Yaote Chang[4] and the extension of this results has been given in [5]. For stochastic systems, most of the results focus on the SISO systems[6]. However, very few results exist for the MIMO stochastic systems with block backstepping design.

For MIMO stochastic systems, the randomness of steady outputs can be bounded once the stochastic systems are stabilized. However, the randomness of the systems outputs would affect the control performance subjected to the random noises and couplings among the stochastic outputs[7]. Therefore, the investigation of the randomness is significant to enhance the performance of the MIMO stochastic systems. Notice that the probability density functions of these outputs obey non-Gaussian distributions; hence the analysis using variance and covariance are not suitable to this problem. In this case, the entropy criterion is introduced to characterise the randomness of the system outputs. In other words, the purpose of this paper is to stabilize the stochastic systems and minimize the entropy of the system outputs due to the fact that there is no existing solution to attenuate the randomness of a MIMO Semi-linear stochastic system.

Motivated by the block backstepping design, the output feedback stabilization for a class of MIMO semi-linear stochastic systems are investigated in the paper due to the fact that the semi-linear stochastic systems become a significant research topic[8] . Based upon the system model and the control objective, a novel observer-based output feedback controller is design by m-block backstepping which can be used to stabilized the MIMO stochastic systems. Furthermore, the controller design parameters can be optimised by minimizing the entropy criteria, where the entropy of the system outputs can be estimated by multidimensional kernel density estimation. Following this presented control algorithm, the stability of the closed-loop systems can be guaranteed in probability sense while the randomness of the system outputs are attenuated by the parametric optimisation.

II. Preliminaries

A. Problem Description

Consider the following MIMO semi-linear stochastic systems with m blocks which can be formulated as follows:

$$\begin{aligned} d\bar{x}_i &= \left(A_i\bar{x}_i + \bar{x}_{i+1}\right)dt + G_1\left(\bar{x}_1\right)d\beta_t, i = 1, \cdots, m-1 \\ d\bar{x}_m &= \left(A_m\bar{x}_m + \bar{u}\right)dt + G_m\left(\bar{x}_1\right)d\beta_t \\ \bar{y} &= \bar{x}_1 \end{aligned} \quad (1)$$

where β_t is the s-dimensional vector-valued Wiener process, $\bar{x}_i$ is the n-dimensional state vector for i-th block, A_i stands for the coefficient matrices with appropriate dimension, $G_i\left(\cdot\right)$ are n-dimensional nonlinear function. $\bar{y}$ and $\bar{u}$ are the system output vector and the vector-valued control input, respectively. The underlying probability space is triple $(\Omega, \mathscr{F}, \mathrm{P})$, where Ω is the sample space of continuous functions, $\mathscr{F}$ is a filtration adapted to the Wiener process β_t, and P is the reference probability measure on Ω

Note that the system described above is in block strict-feedback format where the system outputs and control inputs are with the same dimension. As mentioned in Section I, the control objective is to stabilize this stochastic system in probability sense and minimize the entropy of the stochastic outputs. Before presenting the control algorithm, an assumption about the nonlinear function $G_i(\cdot)$ is given as follows:

Assumption 1: The $n \times s$ nonlinear function $G_i(\cdot)$ for each block of the semi-linear stochastic systems(1) satisfies

$$\|G_i(X_i)\|_2 \leq \sigma_i \tag{2}$$

where $\|\cdot\|_2$ denotes the induced Euclidean norm for matrices and σ_i is one positive real constant.

B. Stability in Probability Sense

Consider the general stochastic nonlinear system:

$$dx = f(x)\,dt + g(x)\,dw \tag{3}$$

where $x \in \mathbb{R}^n$ is the state, w is an r-dimensional independent standard Wiener process, the underlying probability space is the triple $(\Omega, \mathscr{F}, \mathrm{P})$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are locally Lipschitzian and with the following initial values

$$f(0) = 0, g(0) = 0 \tag{4}$$

Definition 1 ([9]): The solution process $\{x(t), t \geq 0\}$ of the stochastic system (3) is said to be bounded in probability if $\lim\limits_{t\to\infty} \sup\limits_{0\leq c\leq\infty} P\{|x(t)| > c\} = 0$

Definition 2 ([10]): For any given $V(x) \in \mathscr{C}^{1,2}$, associated with the stochastic differential equation (3), the differential operator $\mathscr{L}$ can be defined as follows:

$$\mathscr{L}V = \frac{\partial V}{\partial x} f(x) + \frac{1}{2} Tr\left\{g^T(x)\frac{\partial^2 V}{\partial x^2} g(x)\right\} \tag{5}$$

We recall the following lemma [6] which gives the sufficient conditions on the boundedness in probability sense.

Lemma 1 ([6]): Consider system (3) and suppose that there exists a positive-define and radially unbounded function $V(x) \in \mathscr{C}^{1,2}$,$\mu_1(\cdot), \mu_2(\cdot) \in \mathscr{K}_\infty$, positive-define and radially unbounded function $W(x)$ and constant $c > 0$ such that

$$\begin{aligned} &\mu_1(|x|) \leq V(x) \leq \mu_2(|x|) \\ &\mathscr{L}V(x) \leq -W(x) + c \end{aligned} \tag{6}$$

then the solution process of the system (3) is bounded in probability sense.

C. Entropy and Kernel Density Estimation

The information theory has been introduced by [11], where entropy can be used as a measure of the uncertainty of the random variables. For various purposes, a lot of different definitions of the entropy have been presented such as Shannon entropy, Min-entropy, Hartley entropy and Rényi's entropy. In this paper, the quadratic Rényi's entropy is selected to use, which has been introduced by [12] with the form

$$H_2(\bar{y}) = -\log \int \gamma^2(\bar{y}) d\bar{y} \tag{7}$$

where $\gamma(\cdot)$ stands for the joint probability density functions (JPDF) of the system outputs. The data-based multidimensional kernel density estimation (MKDE) [13] is used to estimate the JPDF of the random variables. Note that entropy is equivalent to variance for Gaussian variable [14].

For vector-valued continuous system outputs $\bar{y} \in \mathbb{R}^n$, with its sampled data points $\{\bar{y}_k : k = 1, \ldots, N\}$, the probability density function system output $\bar{y}$ can be estimated as follows:

$$\hat{\gamma}(\bar{y}) = \frac{1}{N}\sum_{k=1}^{N} G_\Sigma(\bar{y} - \bar{y}_k) \tag{8}$$

where $G_\Sigma(\cdot)$ is the Gaussian function defined as follows:

$$G_\Sigma(x) = (2\pi)^{-\frac{n}{2}}(\det \Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}x^T\Sigma^{-1}x\right) \tag{9}$$

Since the JPDF can be estimated by MKDE, Eq.(7) can be rewritten as follows:

$$H_2(\bar{y}) = -\log V(\bar{y}) \tag{10}$$

where $V(\cdot)$ stands for information potential [11]. Furthermore, it can be approximated by

$$\hat{V}(\bar{y}) = \frac{1}{N^2}\sum_{i,j=1}^{N} G_{\sqrt{2\Sigma}}(\bar{y}_i - \bar{y}_j) \tag{11}$$

III. Control Algorithm

A. Linear estimator design

The estimator can be designed as follows.

$$\begin{aligned} d\hat{\bar{x}}_i &= \left(A_i\hat{\bar{x}}_i + \hat{\bar{x}}_{i+1} + L_i\left(\bar{y} - \hat{\bar{x}}_1\right)\right)dt, i = 1, \cdots, m-1 \\ d\hat{\bar{x}}_m &= \left(A_m\hat{\bar{x}}_m + \bar{u} + L_m\left(\bar{y} - \hat{\bar{x}}_1\right)\right)dt \end{aligned} \tag{12}$$

where L_i is the gain of estimator.

The error of the estimator can be introduced as $\tilde{\bar{x}} = \bar{x} - \hat{\bar{x}}$ which satisfies

$$\begin{aligned} d\tilde{\bar{x}} &= \begin{bmatrix} A_1 - L_1 & I & & \\ -L_2 & A_2 & \ddots & \\ \vdots & \vdots & \ddots & I \\ -L_m & 0 & \ldots & A_m \end{bmatrix} \tilde{\bar{x}}dt + \begin{bmatrix} G_1(\bar{y}) \\ G_2(\bar{y}) \\ \vdots \\ G_m(\bar{y}) \end{bmatrix} d\beta_t \\ &:= A_0\tilde{\bar{x}}dt + G_0(\bar{y})\,d\beta_t \end{aligned} \tag{13}$$

Using the linear observer design method, A_0 should be designed to be Hurwitz, then the semi-linear stochastic system with estimator can be re-expressed as follows:

$$\begin{aligned} d\tilde{\bar{x}} &= A_0\tilde{\bar{x}}dt + G_0(\bar{y})\,d\beta_t \\ d\bar{x}_1 &= \left(A_1\bar{x}_1 + \tilde{\bar{x}}_2 + \hat{\bar{x}}_2\right)dt + G_1(\bar{y})\,d\beta_t \\ d\hat{\bar{x}}_i &= \left(F_i + \hat{\bar{x}}_{i+1}\right)dt, i = 2, \cdots, m-1 \\ d\hat{\bar{x}}_m &= \left(F_m + \bar{u}\right)dt \\ \bar{y} &= \bar{x}_1 \end{aligned} \tag{14}$$

where $F_i = A_i\hat{\bar{x}}_i + L_i\left(\bar{y} - \hat{\bar{x}}_1\right), i = 2, \cdots, m$

B. Block backstepping controller design

Since the semi-linear stochastic system with linear estimator is in the strict feedback form, block backstepping can be applied for this MIMO case.

For each block of the original system(1), consider $\bar{\varphi}_i\left(\bar{y},\hat{x}_i\right)$ as the virtual input which can be rewritten as

$$\bar{\varphi}_i\left(\bar{y},\hat{x}_i\right)=\left[\varphi_{i1}\left(\bar{y},\hat{x}_i\right),\ldots,\varphi_{in}\left(\bar{y},\hat{x}_i\right)\right]^T \tag{15}$$

where $\hat{x}_i=[\bar{x}_1,\hat{\bar{x}}_2,\ldots,\hat{\bar{x}}_i]$, $i=1,\cdots,m-1$.

Then the vector-valued error variables can be presented by

$$\bar{z}_i=\hat{\bar{x}}_{i+1}-\bar{\varphi}_i\left(\bar{y},\hat{x}_i\right) \tag{16}$$

where $\bar{z}_i=\left[z_{i1},\ldots,z_{in}\right]^T$,$i=1,\cdots,m-1$.

Using Ito's lemma, we have

$$\begin{aligned}d\bar{z}_i&=\Bigg[\left(F_{i+1}+\hat{\bar{x}}_{i+2}\right)-\Phi_1\left(F_1+\hat{\bar{x}}_2\right)-\frac{1}{2}\Pi_1\\&\quad-\sum_{l=2}^{i}\Phi_l\left(F_l+\hat{\bar{x}}_{l+1}\right)\Bigg]dt-\Phi_1G_1\left(\bar{y}\right)d\beta_t\\&=\left(\Xi_i+\hat{\bar{x}}_{i+2}\right)dt-\Phi_1G_1\left(\bar{y}\right)d\beta_t\end{aligned} \tag{17}$$

where

$$\begin{aligned}\Phi_1&=\left[\nabla_{\bar{y}}\varphi_{11}\left(\bar{y}\right),\ldots,\nabla_{\bar{y}}\varphi_{1n}\left(\bar{y}\right)\right]^T\\\Phi_l&=\left[\nabla_{\hat{\bar{x}}_l}\varphi_{l1}\left(\bar{y},\hat{x}_l\right),\ldots,\nabla_{\hat{\bar{x}}_l}\varphi_{ln}\left(\bar{y},\hat{x}_l\right)\right]^T\\\Pi_1&=\begin{bmatrix}Tr\left\{G_1^T\left(\bar{x}\right)\left(H_{\bar{x}}\varphi_{11}\left(\bar{x}_1\right)\right)G_1\left(\bar{x}_1\right)\right\}\\\vdots\\Tr\left\{G_1^T\left(\bar{x}\right)\left(H_{\bar{x}}\varphi_{1n}\left(\bar{x}_1\right)\right)G_1\left(\bar{x}_1\right)\right\}\end{bmatrix}\\\Xi_i&=F_{i+1}-\Phi_1\left(F_1+\hat{\bar{x}}_2\right)-\frac{1}{2}\Pi_1-\sum_{l=2}^{i}\Phi_l\left(F_l+\hat{\bar{x}}_{l+1}\right)\end{aligned} \tag{18}$$

To stabilize the entire stochastic system, we employ a Lyapunov function candidate which is presented as follows:

$$V=\frac{1}{2}\sum_{k=1}^{n}\bar{y}_k^2+\frac{b}{2}\left(\tilde{x}^TP\tilde{x}\right)^2+\frac{1}{4}\sum_{i=1}^{m-1}\sum_{l=1}^{n}\bar{z}_{il}^4 \tag{19}$$

where P denotes the positive definite matrix which satisfies ${A_0}^TP+PA_0<0$.

Now we start to analyze the property of $\mathscr{L}V$ for the proposed Lypunov function candidate.

$$\begin{aligned}\mathscr{L}V&=\bar{y}^T\left(A_1\bar{y}+\tilde{\bar{x}}_2+\hat{\bar{x}}_2\right)+\frac{1}{2}Tr\left\{G_1^T\left(\bar{y}\right)G_1\left(\bar{y}\right)\right\}\\&\quad+\sum_{i=1}^{m-1}\eta_i\left(\Xi_i+\hat{\bar{x}}_{i+2}\right)+\frac{3}{2}Tr\left\{\Gamma_i\Phi_1G_1\left(\bar{y}\right)\left(\Phi_1G_1\left(\bar{y}\right)\right)^T\right\}\\&\quad+2bTr\left\{G_0^T\left(\bar{y}\right)\left(2P\tilde{x}\tilde{x}^TP+\tilde{x}^TP\tilde{x}P\right)G_0\left(\bar{y}\right)\right\}\\&\quad-b\tilde{x}^TP\tilde{x}\|\tilde{x}\|^2\end{aligned} \tag{20}$$

where

$$\begin{aligned}\eta_i&=\left[z_{i1}^3,\ldots,z_{in}^3\right]\\\Gamma_i&=diag\left(z_{i1}^2,\ldots,z_{in}^2\right)\end{aligned} \tag{21}$$

The trace terms of $\mathscr{L}V$ remain difficult to handle, thus a useful lemma is given here which can be used repeatedly to simplify the formulation.

Lemma 2: Consider that $A_1,A_2,B\in R^{n\times n}$ are n-dimensional square matrices and $D\in R^{n\times n}$ is diagonal matrix, where $A_1=\left[\bar{a}_{11},\ldots,\bar{a}_{1n}\right]^T$, $A_2=\left[\bar{a}_{21},\ldots,\bar{a}_{2n}\right]$ and $D=diag\left\{d_1,\ldots,d_n\right\}$. Then the following inequality holds.

$$Tr\left\{DA_1BA_2\right\}\le\sum_{i=1}^{n}\|d_i\|\|\bar{a}_{1i}\|\|\bar{a}_{2i}\|\|B\| \tag{22}$$

Proof: Using the structure of the matrices which has been mentioned above, we can have

$$\begin{aligned}&Tr\left\{DA_1BA_2\right\}\\&=Tr\left\{\begin{bmatrix}d_1&&\\&\ddots&\\&&d_n\end{bmatrix}\begin{bmatrix}a_{11}\\\vdots\\a_{1n}\end{bmatrix}B\begin{bmatrix}a_{21}&\cdots&a_{2n}\end{bmatrix}\right\}\\&=Tr\left\{\begin{bmatrix}d_1&&\\&\ddots&\\&&d_n\end{bmatrix}\begin{bmatrix}a_{11}Ba_{21}&\ldots&a_{11}Ba_{2n}\\\vdots&\ddots&\vdots\\a_{1n}Ba_{21}&\ldots&a_{1n}Ba_{2n}\end{bmatrix}\right\}\\&=\sum_{i=1}^{n}d_ia_{1i}Ba_{2i}\\&\le\sum_{i=1}^{n}\|d_ia_{1i}Ba_{2i}\|\end{aligned} \tag{23}$$

Following the property of the norm operation, we have

$$\sum_{i=1}^{n}\|d_ia_{1i}Ba_{2i}\|\le\sum_{i=1}^{n}\|d_i\|\|a_{1i}\|\|a_{2i}\|\|B\| \tag{24}$$

which ends the proof of Lemma 2 ■

Based upon Lemma 2, we can obtain the following inequalities in order to deal with the trace terms in Eq.(20) separately using Young's inequality.

At firstly, we have

$$\begin{aligned}&Tr\left\{\Gamma\Phi_1G_1\left(\bar{y}\right)G_1^T\left(\bar{y}\right)\Phi_1^T\right\}\\&\quad\le\sum_{i=1}^{n}z_{1i}^2\left\|\nabla_{\bar{x}}\varphi_{1i}^T\left(\bar{y}\right)\right\|_2\|\nabla_{\bar{x}}\varphi_{1i}\left(\bar{y}\right)\|_2\|G_1\left(\bar{y}\right)\|_2\left\|G_1^T\left(\bar{y}\right)\right\|_2\\&\quad\le\sum_{i=1}^{n}\frac{\varepsilon_{1i}^2}{2}z_{1i}^4\left\|\nabla_{\bar{x}}\varphi_{1i}^T\left(\bar{y}\right)\right\|^2+\sum_{i=1}^{n}\frac{1}{2\varepsilon_{1i}^2}\|G_1\left(\bar{y}\right)\|_2^4\\&\quad=\sum_{i=1}^{n}\frac{\varepsilon_{1i}^2}{2}z_{1i}^4\left\|\nabla_{\bar{x}}\varphi_{1i}^T\left(\bar{x}_1\right)\right\|^2+\sum_{i=1}^{n}\frac{1}{2\varepsilon_{1i}^2}\sigma_1^4\end{aligned} \tag{25}$$

Next, we further obtain

$$Tr\left\{G_1^T\left(\bar{y}\right)G_1\left(\bar{y}\right)\right\}\le n\|G_1\left(\bar{y}\right)\|^2=n\sigma_1^2 \tag{26}$$

and

$$
\begin{aligned}
&2bTr\left\{G_0^T(\bar{y})\left(2P\tilde{x}\tilde{x}^TP+\tilde{x}^TP\tilde{x}P\right)G_0(\bar{y})\right\}\\
&\quad=4b\left\|G_0^T(\bar{y})P\tilde{x}\right\|_F^2+2b\tilde{x}^TP\tilde{x}Tr\left\{PG_0(\bar{y})G_0^T(\bar{y})\right\}\\
&\quad\leq 4b\sqrt{s}\left\|G_0^T(\bar{y})\right\|^2\|P\|^2\|\tilde{x}\|^2\\
&\quad+2b\left\|Tr\left\{PG_0(\bar{y})G_0^T(\bar{y})\right\}\right\|\|P\|\|\tilde{x}\|^2\\
&\quad\leq\left(\frac{\tilde{\varepsilon}_1^2+\tilde{\varepsilon}_2^2}{2}\right)\|\tilde{x}\|^4+\frac{8b^2s}{\tilde{\varepsilon}_1^2}\left\|G_0^T(\bar{y})\right\|^4\|P\|^4\\
&\quad+\frac{2b^2}{\tilde{\varepsilon}_2^2}\|P\|^2\left(\sum_{i=1}^{nm}|p_i|\left\|G_0^T(\bar{y})\right\|^2\right)^2\\
&\quad=\left(\frac{\tilde{\varepsilon}_1^2+\tilde{\varepsilon}_2^2}{2}\right)\|\tilde{x}\|^4+\tilde{c}
\end{aligned}\tag{27}
$$

where $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$, ε_{1i} are any real positive numbers. $\|\cdot\|_F$ stands for Frobenius norm.

Moreover, notice that $\tilde{c}\geq 0$,

$$
\tilde{c}=\frac{8b^2s}{\tilde{\varepsilon}_1^2}\left\|G_0^T(\bar{y})\right\|^4\|P\|^4+\frac{2b^2}{\tilde{\varepsilon}_2^2}\|P\|^2\left(\sum_{i=1}^{nm}|p_i|\left\|G_0^T(\bar{y})\right\|^2\right)^2\tag{28}
$$

and the following inequality always holds

$$
-b\tilde{x}^TP\tilde{x}\|\tilde{x}\|^2\leq -b\lambda_{\min}\{P\}\|\tilde{x}\|^4\tag{29}
$$

Substituting these inequalities Eq. (23)-Eq. (26) to $\mathscr{L}V$, as a result, Eq. (20) can be rewritten as the following inequality which can be used to evaluate the controller for stabilization.

$$
\begin{aligned}
&\mathscr{L}V\leq\bar{y}^T\left(A_1\bar{y}+\tilde{\bar{x}}_2+\hat{\bar{x}}_2\right)+\frac{1}{2}n\sigma_1^2+\sum_{i=1}^{m-1}\eta_i\left(\Xi_i+\hat{\bar{x}}_{i+2}\right)\\
&+\frac{3}{2}\left(\sum_{i=1}^{n}\frac{\varepsilon_{1i}^2}{2}z_{1i}^4\left\|\nabla_{\bar{x}}\varphi_{1i}^T(\bar{x}_1)\right\|^2+\sum_{i=1}^{n}\frac{1}{2\varepsilon_{1i}^2}\sigma_1^4\right)\\
&-b\lambda_{\min}\{P\}\|\tilde{x}\|^4+\left(\frac{\tilde{\varepsilon}_1^2+\tilde{\varepsilon}_2^2}{2}\right)\|\tilde{x}\|^4+\tilde{c}\\
&=\bar{y}^T\left(A_1\bar{y}+\tilde{\bar{x}}_2+\hat{\bar{x}}_2\right)\\
&+\sum_{i=2}^{m-1}\eta_i\left(\Xi_i+\bar{z}_{i+1}+\bar{\varphi}_{i+1}(\bar{y},\hat{x}_{i+1})\right)\\
&+\eta_1\left(\Xi_1+\bar{z}_2+\bar{\varphi}_2\left(\bar{y},\hat{\bar{x}}_2\right)+\begin{bmatrix}\frac{\varepsilon_{11}^2}{2}z_{11}\left\|\nabla_{\bar{x}}\varphi_{11}^T(\bar{y})\right\|^2\\ \vdots\\ \frac{\varepsilon_{1n}^2}{2}z_{1n}\left\|\nabla_{\bar{x}}\varphi_{1n}^T(\bar{y})\right\|^2\end{bmatrix}\right)\\
&+\frac{3}{2}\sum_{i=1}^{n}\frac{1}{2\varepsilon_{1i}^2}\sigma_1^4+\tilde{c}-\left(b\lambda_{\min}\{P\}-\frac{\tilde{\varepsilon}_1^2+\tilde{\varepsilon}_2^2}{2}\right)\|\tilde{x}\|^4
\end{aligned}\tag{30}
$$

Based upon Lemma 1, the control input and virtual inputs can be chosen as

$$
\begin{aligned}
\bar{\varphi}_1(\bar{y})&=(-W-A_1)\bar{y}-\tilde{\bar{x}}_2\\
\bar{\varphi}_2\left(\bar{y},\hat{\bar{x}}_2\right)&=-\Xi_1-\bar{z}_2-\Theta-C_1\bar{z}_1\\
\bar{\varphi}_{i+1}(\bar{y},\hat{x}_{i+1})&=-\Xi_i-\bar{z}_{i+1}-C_i\bar{z}_i\\
u=\bar{\varphi}_m(\bar{y},\hat{x}_m)&=-\Xi_{m-1}-\bar{z}_m-C_{m-1}\bar{z}_{m-1}
\end{aligned}\tag{31}
$$

where W denotes the positive definite matrix.

$$
\Theta=\begin{bmatrix}\frac{\varepsilon_{11}^2}{2}z_{11}\left\|\nabla_{\bar{x}}\varphi_{11}^T(\bar{y})\right\|^2\\ \vdots\\ \frac{\varepsilon_{1n}^2}{2}z_{1n}\left\|\nabla_{\bar{x}}\varphi_{1n}^T(\bar{y})\right\|^2\end{bmatrix}
$$

$$
C_i=diag\left[c_{i1},\ldots,c_{in}\right],c_{ij}>0\tag{32}
$$

Furthermore, $\mathscr{L}V$ can be rewritten as

$$
\mathscr{L}V=-\bar{y}^TW\bar{y}-\tilde{p}\|\tilde{x}\|^4-\sum_{i=1}^{m-1}\sum_{l=1}^{n}c_{il}\bar{z}_{il}^4+\bar{c}\tag{33}
$$

where

$$
\tilde{p}=b\lambda_{\min}\{P\}-\frac{\tilde{\varepsilon}_1^2+\tilde{\varepsilon}_2^2}{2},\bar{c}=\frac{3}{2}\sum_{i=1}^{n}\frac{1}{2\varepsilon_{1i}^2}\sigma_1^4+\tilde{c}\tag{34}
$$

Thus, the stabilisation of the closed-loop semi-linear stochastic system is analysed by the following theorem in which the structure of the controller that can stabilise the presented closed-loop stochastic system.

Theorem 1: The semi-linear stochastic system (1) with linear estimator (12) and control law (31) are guaranteed to be bounded in probability sense if there exists a positive definite matrix P which makes $\tilde{p}>0$ and ${A_0}^TP+PA_0<0$.

Proof: Since A_0, P and $\tilde{p}$ are defined in Equations (13),(19) and (34), the proof is showed above. ■

C. Output Randomness Optimization

As mentioned in Section II, the performance criterion can be given as follows:

$$
J_k=-\log\hat{V}(\bar{y},W_0)\tag{35}
$$

where W_0 denotes the set of design parameters, $\mathrm{W}_0=\{W,\tilde{\varepsilon}_1,\tilde{\varepsilon}_2,\varepsilon_{1i},C_i\},i=1,\ldots,n$. k stands for the sampling instant. Since the $\log(\cdot)$function is with the monotonic increasing property, minimizing of the entropy is equivalent to maximizing the information potential, thus the performance criterion can be further simplified only using $\hat{V}(\bar{y},W_0)$.

In addition, the following theorem is given to claim that this performance criterion is a globally convex function based on another assumption.

Assumption 2: The closed-loop stochastic system output vector $\bar{y}$ satisfies the following inequality:

$$
\frac{\partial\bar{y}}{\partial W_0}\leq M\tag{36}
$$

where the real positive matrix M denotes the upper bound.

Theorem 2: For the presneted control algorithm, there exists a real positive number $\delta_0>0$, such that the information potential is globally concave with respect to the design parameter W_0 for all $\lambda_{\min}(\Sigma)>\delta_0$. Thus the equivalent performance criterion (35) is convex with a global optimum.

Proof: Denote $\varepsilon_{ij,k} = \bar{y}_{i,k} - \bar{y}_{j,k}$, then we have

$$\frac{\partial^2 \hat{V}_k (W_0)}{\partial W_0^2} = \frac{1}{N^2} \frac{\partial}{\partial W_0} \sum_{i,j=1}^{N} \frac{\partial}{\partial W_0} G_{\sqrt{2}\Sigma} (\varepsilon_{ij,k})$$
$$= \frac{1}{N^2} \frac{\partial}{\partial W_0} \sum_{i,j=1}^{N} \frac{\partial G_{\sqrt{2}\Sigma} (\varepsilon_{ij,k})}{\partial \varepsilon_{ij,k}} \frac{\partial \varepsilon_{ij,k}}{\partial W_0}$$
$$\leq -\frac{1}{N^2} \left(\sqrt{2}\Sigma\right)^{-1} \frac{\partial}{\partial W_0} \sum_{i,j=1}^{N} G_{\sqrt{2}\Sigma} (\varepsilon_{ij,k}) \times \varepsilon_{ij,k} M$$
$$= -\frac{M}{N^2} \left(\sqrt{2}\Sigma\right)^{-1} \sum_{i,j=1}^{N} G_{\sqrt{2}\Sigma} (\varepsilon_{ij,k})$$
$$\times \left(\varepsilon_{ij,k}^T \left(M - \left(\sqrt{2}\Sigma\right)^{-1}\right) \varepsilon_{ij,k}\right) \quad (37)$$

As a result, $\frac{\partial^2 V_k(\varepsilon)}{\partial W_0^2} \leq 0$ if $M \geq \left(\sqrt{2}\Sigma\right)^{-1}$. It is shown that the eigenvalues of $\frac{\partial^2 V_k(\varepsilon)}{\partial W_0^2}$ approach 0^- as $\lambda_{\min}(\Sigma)$ goes to infinity. Based on the Lemma 3 in [15], $\hat{V}_k(\bar{y}, W_0)$ will be concave since $\lambda_{\min}(\Sigma)$ is sufficiently large. Moreover, the performance criterion (35) is convex which results in the global optimum. It shows that the proof is completed. ■

Since the performance criterion is convex then the standard convex optimization approach can be applied to this issue directly. Without loss of generality, the gradient descent optimization is given as follows:

$$W_{0,k+1} = W_{0,k} - \gamma \left. \frac{\partial \hat{V}_k (W_0)}{\partial W_0} \right|_{W_0 = W_{0,k}} \quad (38)$$

where $\gamma > 0$ denotes the pre-specified step.

IV. An Numerical Example

To demonstrate the presented algorithm procedure, a multi-variable semi-linear stochastic system is shown as follows:

$$d\bar{x}_1 = \left(\begin{bmatrix} -1 & 0.5 \\ 0 & -2 \end{bmatrix} \bar{x}_1 + \bar{x}_2 \right) dt + sin(\bar{x}_1) d\beta_t$$
$$d\bar{x}_2 = \left(\begin{bmatrix} -1.5 & 0 \\ -0.5 & -1 \end{bmatrix} \bar{x}_2 + \bar{u} \right) dt + cos(\bar{x}_1) d\beta_t$$
$$\bar{y} = \bar{x}_1 \quad (39)$$

where A_1, A_2, G_1 and G_2 are given as coefficient matrices.

In addition, the linear estimator can be obtained with the feedback gain matrices $L_1 = L_2 = diag\{15, 15\}$. Thus, the closed-loop system with estimator can be rewritten by

$$d\bar{y} = \left(\begin{bmatrix} -1 & 0.5 \\ 0 & -2 \end{bmatrix} \bar{y} + \tilde{\bar{x}}_2 + \hat{\bar{x}}_2 \right) dt + \sin(\bar{y}) d\beta_t$$
$$d\hat{\bar{x}}_2 = \left(\begin{bmatrix} -1.5 & 0 \\ -0.5 & -1 \end{bmatrix} \hat{\bar{x}}_2 + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \tilde{\bar{x}}_1 + \bar{u} \right) dt$$
$$d \begin{bmatrix} \tilde{\bar{x}}_1 \\ \tilde{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} -3 & 0.5 & 1 & 0 \\ 0 & -5 & 0 & 1 \\ -2 & 0 & -1.5 & 0 \\ 0 & -3 & -0.5 & -1 \end{bmatrix} \begin{bmatrix} \tilde{\bar{x}}_1 \\ \tilde{\bar{x}}_2 \end{bmatrix} dt$$
$$+ \begin{bmatrix} \sin(\bar{y}) \\ \cos(\bar{y}) \end{bmatrix} d\beta_t \quad (40)$$

where A_0 and G_0 are obtained with A_0 being Hurwitz.

As we mentioned in section III, the first virtual control input and the control input can be designed using Eq.(31). It has been shown that the design parameters W_0 will affect the performance of the controller, then the optimisation is essential while $W = diag\{-20, 25\}$ can be pre-selected and other parameters can be initialized by positive random numbers.

In this simulation, the sampling time k is selected as $0.01s$, and the control performance of the closed-loop stochastic system are given by curves in Figs. 1-3. The output trajectories are indicated by Fig.1. In particular, the stochastic outputs have been stabilized rapidly. The control input is depicted by Fig.2. In Fig.3, the performance criterion are given using KDE which results in a smooth curve. It shows that the value of J descends when the presented optimisation method searches of the optimal design parameters. Simultaneously, the randomness of the system outputs has been minimised. Notice that $G_2(0) \neq 0$, thus the output trajectories are just bounded in probability sense. If we change the model as $G_2(\bar{y}) = sin(\bar{y})$, then the output will convergent to 0 in probability sense and the performance can be shown by Fig. 4

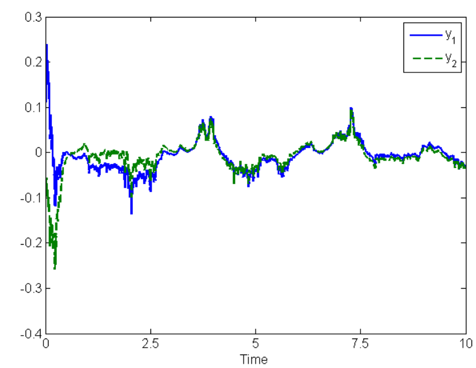

Fig. 1. Output trajectories of the closed-loop stochastic system

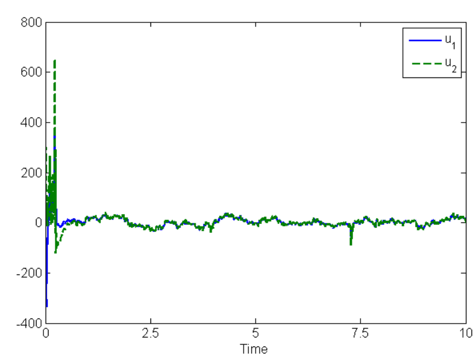

Fig. 2. The control input signal

V. Further Discussion

The presented control algorithm is very convenient to extend to bilinear stochastic systems [10] which implies that Assumption 1 in this paper can be released to the Lipschitz condition.

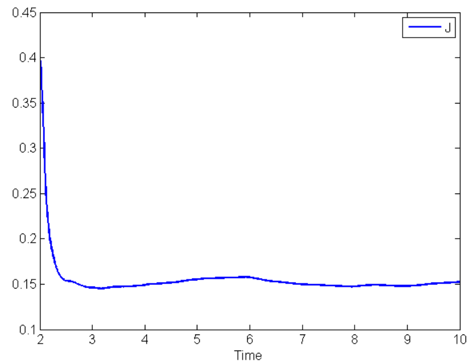

Fig. 3. The value of performance criteria J

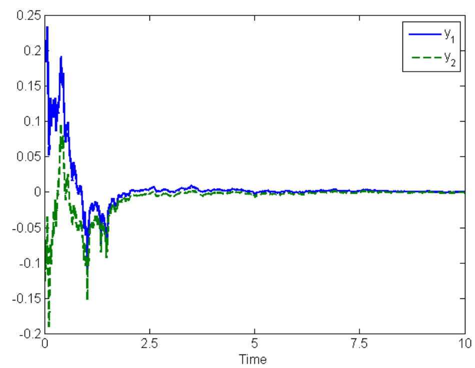

Fig. 4. Output trajectories of the control stochastic system with $G_2(0) = 0$

Then the outputs of the stochastic systems can be stabilized using block backstepping design while the performance of the systems can be enhanced by optimizing the design parameters. Naturally, not only the randomness attenuation problem can be investigated but also other control objectives can be considered following various performance criteria. Therefore, we can extend this control algorithm as a framework of performance enhancement for a class of dynamic stochastic systems.

Particularly, the couplings among the system outputs can also be investigated using this presented framework. Based on the concept of probabilistic decoupling, the performance criterion can be described using probability density functions of the system outputs as follows.

$$J_k = \min_{W_0} \left\| \gamma\left(\bar{y}\right) - \gamma_i\left(y_i\right) \right\| \tag{41}$$

where $\gamma_i(\cdot)$ stands for the probability density function (PDF) of each system outputs which can also be approximated by kernel density estimation (KDE).

VI. Conclusions

For a class of MIMO semi-linear stochastic systems, a output feedback control algorithm is presented for the problem of transient randomness attenuation. Based on the linear estimator and the m-block backstepping design, the structure of the controller is obtained to stabilize the closed-loop stochastic system. Since the design parameters of the controller affect the performance of the system outputs, the minimum entropy performance criterion is given to attenuate the output randomness of the closed-loop system. Furthermore, the performance criterion can be restated using information potential equivalently. Based upon the multidimensional kernel density estimation, the convexity of the presented performance criteria can be guaranteed and standard convex optimization can be used to search the optimal parameters in this case. To verify the presented control algorithm, a numerical example is given while the simulation results show the effectiveness. In addition, this control algorithm can be considered as a framework because it can be extended to bilinear or even Lipschitz non-linear stochastic system and other performance criteria can also be considered by the similar approach, such as probabilistic decoupling, etc.

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