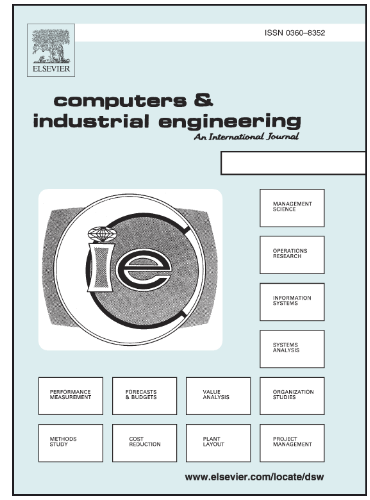
Accepted Manuscript



Data Envelopment Analysis Models with Ratio Data: A revisit

Adel Hatami-Marbini, Mehdi Toloo

PII: S0360-8352(19)30252-9
DOI: https://doi.org/10.1016/j.cie.2019.04.041
Reference: CAIE 5828

To appear in: *Computers & Industrial Engineering*

Received Date: 26 November 2018
Revised Date: 16 March 2019
Accepted Date: 23 April 2019

Please cite this article as: Hatami-Marbini, A., Toloo, M., Data Envelopment Analysis Models with Ratio Data: A revisit, *Computers & Industrial Engineering* (2019), doi: https://doi.org/10.1016/j.cie.2019.04.041

Data Envelopment Analysis Models with Ratio Data: A revisit

Adel Hatami-Marbini*
Department of Strategic Management and Marketing
Leicester Castle Business School, De Montfort University
Hugh Aston Building, The Gateway, Leicester LE1 9BH, UK
E-mail: adel.hatamimarbini@dmu.ac.uk

Mehdi Toloo
Department of Systems Engineering, Faculty of Economics,
Technical University of Ostrava, Sokolská třída 33, 702 00 Ostrava 1,
Ostrava, Czech Republic
E-mail: mehdi.toloo@vsb.cz
URL: http://homel.vsb.cz/~tol0013/

Acknowledgement
The authors would like to thank the anonymous reviewers and the editor for their insightful comments and suggestions. Dr. Mehdi Toloo is grateful for the support he received from the Czech Science Foundation (GAČR 17-23495S) for this research.

* Corresponding author

Data Envelopment Analysis Models with Ratio Data: A revisit

Abstract

The performance evaluation of for-profit and not-for-profit organisations is a unique tool to support the continuous improvement process. Data envelopment analysis (DEA) is literally known as an impeccable technique for efficiency measurement. However, the lack of the ability to attend to ratio measures is an ongoing challenge in DEA. The convexity axiom embedded in standard DEA models cannot be fully satisfied where the dataset includes ratio measures and the results obtained from such models may not be correct and reliable. There is a typical approach to deal with the problem of ratio measures in DEA, in particular when numerators and denominators of ratio data are available. In this paper, we show that the current solutions may also fail to preserve the principal properties of DEA as well as to instigate some other flaws. We also make modifications to explicitly overcome the flaws and measure the performance of a set of operating units for the input- and output orientations regardless of assumed technology. Finally, a case study in the education sector is presented to illustrate the strengths and limitations of the proposed approach.

Keywords: Data envelopment analysis; Ratio measures; Efficiency measure; Technology.

1. Introduction

Analysing and managing the performance of organisations is a key responsibility of the top-level management team that can be carried out by different techniques. Data envelopment analysis (DEA) originated by Charnes, Cooper, & Rhodes (1978) is recognised as a successful tool in measuring the relative efficiency of organisations.

In a great variety of applications, the consideration of ratios such as financial ratio data in addition to absolute data for assessing the performance of organisations is undeniable. This regard may be essential to estimate the underlying production frontier in DEA. Mathematically speaking, a ratio (A/B) is a quantitative relationship between two numbers, A and B, showing the number of times A contains within B. To tackle ratio measures, Thanassoulis, Boussofiane, & Dyson (1995) use an approach where ratio measures are replaced by absolute measures.

Hollingsworth & Smith (2003) originally discuss that, in the presence of ratio measures, the results of the original DEA models may not be correct and acceptable thanks to being inconsistent with the production axioms. To address and argue this imperative problem, the relevant literature can be divided into two main streams. The first stream aims to deal efficiently with the situations where the numerator and denominator of each ratio measure are known (Emrouznejad & Amin, 2009). Emrouznejad & Amin (2009) propose the input- and output-orientated DEA models by revisiting the axioms to deal with ratio measures. Khoshnevis & Teirlinck (2018) evaluate the efficiency of R&D active firms in Belgium using Emrouznejad & Amin (2009)'s approach in which *net added value per employee, turnover per employee* and *R&D intensity* are ratio measures. Of late, Gidion, Hong, Adams, & Khoveyni (2019) argue that urban water utilities performance involves multiple scaled weight components which utilise key performance indicators as ratio input in the yardstick competition regime. They formulate a network DEA model with ratio data according to Emrouznejad & Amin (2009) to evaluate 40 urban water utilities in Africa, Asia, and Europe.

The main purpose of forming the second stream is to tackle the conditions that the underlying absolute data associated with ratios are not available (Olesen & Petersen, 2009). Olesen, Petersen, & Podinovski (2015) develop a pair of production technologies to present ratio constant returns-to-scale (R-CRS) and ratio variable returns-to-scale (R-VRS) models in which ratio measures are allowed to incorporate into inputs and outputs without data transformation. In what follows, they adapt the traditional production axioms to state the new axioms of convexity and proportionality in the presence of ratio measures. Furthermore, Olesen et al. (2015) classify ratio measures into different groups and argue that the R-CRS technology of each group is modelled differently to respond to the proportional changes in absolute inputs. These ratio groups are *proportional ratios* (those that increase proportionality with the increase in absolute measures), *fixed ratios* (those that do not change when absolute measures change), *downward proportional ratios* (are proportional when absolute measures decrease but fixed when they increase), and *upward proportional ratios* (are proportional when absolute measures increase, but fixed when absolute measures decrease).

Carayannis, Grigoroudis, & Goletsis (2016) propose a multi-objective DEA approach to evaluate innovation systems of 23 European countries and their 185 corresponding regions in the presence of ratio data. Innovation systems are first modelled as two sub-processes; *knowledge production process* (KPP) and *knowledge commercialization process* (KCP), and then the DEA model proposed by Kao & Hwang (2008) is adopted to evaluate the overall and stage regional efficiencies. Due to the fact that several ratio factors such as "*participation percentage in lifelong learning*" and "*percentage of employment in knowledge intensive services/manufacturing*" are used in a ratio form, Carayannis et al. (2016) accommodate the two-process DEA model by Olesen et al. (2015).

Following Olesen et al. (2015), Olesen, Petersen, & Podinovski (2017) introduce the potential ratio inefficiency concept in DEA to show that strong efficient DMUs with ratio measures are plausible to be identified as inefficient DMUs in the presence of absolute data. They employ the R-CRS and R-VRS technologies suggested by Olesen et al. (2015) to formulate input radial, output radial, and non-radial DEA models.

Silva (2018) assess the performance measures of Portuguese courts where ratio measures are available. The author thinks of three variants of linkages between inputs and outputs, which are (*i*) separate assessments; (*ii*) ratios between linked outputs and inputs; and (*iii*) differences between linked outputs and inputs. Regarding ratio measures, the DEA model developed by Olesen et al. (2015) is adapted in which a single input is absolute and all outputs are in the form of ratios.

Not only the employment of ratio measures within applications has been increasingly popping up, but also the literature lacks adequate and deep attention to the existing theory of this area. This encourages us to instil one of the most popular and widely used DEA-models and we only lay great emphasis on Emrouznejad & Amin (2009)'s work placed in the first stream.

Emrouznejad & Amin (2009) (hereafter called EA) treat the problems from the input and output orientations by presenting two different solutions when the data includes output- and/or input- ratio variables. The first solution of EA is dependent on whether a ratio measure is an input or output with the aim of transforming it into the auxiliary input and output that are consumed and produced by each decision-making unit (DMU). More precisely, the numerator and denominator of each input-ratio (output-ratio) measure are considered as an additional output (input) and input (output), respectively. The second solution of EA makes an attempt to re-define the axiom of convexity for ratio measures. The majority of DEA models associated with this solution are nonlinear.

Emrouznejad & Amin (2009) proclaims the weakness of the first solution that is the lack of sufficient discrimination power along with the weakness of the second solution that necessitates deploying a nonlinear programming model in many situations. In this study, we show several flaws in both of the solutions developed by Emrouznejad & Amin (2009) besides the aforesaid problems. The inspection of Emrouznejad & Amin (2009)'s solutions allows us to get supplementary properties of DEA model in the presence of ratio measures. We then introduce the modified multiplier and envelopment DEA models for measuring performance in a way that avoid the problems associated with Emrouznejad & Amin (2009)'s models. We finally present a case study in the education sector to highlight the flaws of the existing models as well as to show the advantage of the models proposed in this study.

The remainder of this study is organised as follow: Section 2 presents a critical discussion of both the solutions of Emrouznejad & Amin (2009). In Section 3, we discuss a premise to make any necessary modifications to the models in order to treat the flaws. Section 4 presents a simple case study in the education sector to illustrate the flaws coupled with the applicability of the proposed models. Finally, we sum up our conclusions in Section 5.

2. A critical discussion on the EA models

The EA method introduces two various solutions to deal with ratios in DEA. In this section, we first present the input and output orientations EA models in the general cases as well as providing some remarks and theorems.

2.1. Solution 1 of EA

Assume that there are n DMUs where each $\text{DMU}_j, j \in J = \{1, \dots n\}$, consumes m inputs $\boldsymbol{x}_j = (\dots, x_{ij}, \dots), i \in I = \{1, \dots, m\}$, to produce s outputs $\boldsymbol{y}_j = (\dots, y_{rj}, \dots)$, $r \in O = \{1, .., s\}$. Furthermore, let $I^R \subseteq I$ and $O^R \subseteq O$ be the sub-index representing ratio inputs and ratio outputs, respectively, and $I^A = I \backslash I^R$ and $O^A = O \backslash O^R$ are the sub-index of absolute inputs and outputs, respectively. It is assumed that $x_{ij}, \forall i \in I^R$ ($y_{rj}, \forall r \in O^R$) is calculated as the numerators $\bar{x}_{ij}$ ($\bar{y}_{rj}$) divided by the denominator $\underline{x}_{ij}$ ($\underline{y}_{rj}$), i.e., $x_{ij} = \frac{\bar{x}_{ij}}{\underline{x}_{ij}}, \forall i \in I^R$ ($y_{rj} = \frac{\bar{y}_{rj}}{\underline{y}_{rj}}, \forall r \in O^R$). EA suggests the following input- and output-oriented models as Solution 1 to treat ratio inputs and outputs:

IN-Sol.1 model

$$\begin{aligned} & \min \theta \\ & \text{s.t.} \\ & \textstyle\sum_{j=1}^n \lambda_j x_{ij} \le \theta x_{io} && i \in I^A \\ & \textstyle\sum_{j=1}^n \lambda_j y_{rj} \ge y_{ro} && r \in O^A \\ & \textstyle\sum_{j=1}^n \lambda_j \bar{x}_{ij} \le \theta \bar{x}_{io} && i \in I^R \\ & \textstyle\sum_{j=1}^n \lambda_j \underline{x}_{ij} \ge \underline{x}_{io} && i \in I^R \\ & \textstyle\sum_{j=1}^n \lambda_j \underline{y}_{rj} \le \theta \underline{y}_{ro} && r \in O^R \\ & \textstyle\sum_{j=1}^n \lambda_j \bar{y}_{rj} \ge \bar{y}_{ro} && r \in O^R \\ & \lambda_j \in \vartheta(\gamma) \end{aligned} \tag{1}$$

OUT-Sol.1 model

$$\begin{aligned} & \max \phi \\ & \text{s.t.} \\ & \textstyle\sum_{j=1}^n \lambda_j x_{ij} \le x_{io} && i \in I^A \\ & \textstyle\sum_{j=1}^n \lambda_j y_{rj} \ge \phi y_{ro} && r \in O^A \\ & \textstyle\sum_{j=1}^n \lambda_j \bar{x}_{ij} \le \bar{x}_{io} && i \in I^R \\ & \textstyle\sum_{j=1}^n \lambda_j \underline{x}_{ij} \ge \phi \underline{x}_{io} && i \in I^R \\ & \textstyle\sum_{j=1}^n \lambda_j \underline{y}_{rj} \le \underline{y}_{ro} && r \in O^R \\ & \textstyle\sum_{j=1}^n \lambda_j \bar{y}_{rj} \ge \phi \bar{y}_{ro} && r \in O^R \\ & \lambda_j \in \vartheta(\gamma) \end{aligned} \tag{2}$$

where $\gamma \in \{CRS, VRS, IRS, DRS\}$ and $\vartheta(CRS) = \{\lambda_j \in \mathbb{R}^n | \lambda_j \ge 0\}$, $\vartheta(VRS) = \{\lambda_j \in \mathbb{R}^n | \sum_{j=1}^n \lambda_j = 1, \lambda_j \ge 0\}$, $\vartheta(IRS) = \{\lambda_j \in \mathbb{R}^n | \sum_{j=1}^n \lambda_j \ge 1, \lambda_j \ge 0\}$, and $\vartheta(DRS) = \{\lambda_j \in \mathbb{R}^n | \sum_{j=1}^n \lambda_j \le 1, \lambda_j \ge 0\}$ represent CRS, VRS, IRS, and DRS models, respectively. In the above models, each input numerator $\bar{x}_{ij}$ and output denominator $\underline{y}_{rj}$ associated with ratio measures play the role of the absolute inputs, and each input denominator $\underline{x}_{ij}$ and output numerator $\bar{y}_{rj}$ play the role of the absolute outputs. In other words, in these models the input and output vectors for DMU_j are $\left(\boldsymbol{x}_j, \bar{\boldsymbol{x}}_j, \underline{\boldsymbol{y}}_j\right) \in \mathbb{R}^{|I^A|+|I^R|+|O^R|}$ and $\left(\boldsymbol{y}_j, \bar{\boldsymbol{y}}_j, \underline{\boldsymbol{x}}_j\right) \in \mathbb{R}^{|O^A|+|O^R|+|I^R|}$, respectively, where $|.|$ denotes the cardinality (number of elements) of a set. Note that though EA also discusses the special cases of models (1) and (2) under the VRS assumption, we concentrate on these general models which allow us to adapt them to different configurations. The discriminatory power of the above models can be reduced thanks to increase in the number of inputs and outputs (Emrouznejad and Amin, 2009).

The following three flaws hinder the applicability of models (1) and (2).

Flaw 1. The purpose of standard input-oriented DEA models based on Farrell measure is to radially reduce the amount of input vector $\boldsymbol{x}_o$ to $\theta^*\boldsymbol{x}_o$ with output vector held fixed while regardless of our expectation, in model (1), the radial reduction in the output numerator $\underline{\boldsymbol{y}}_{ro}, \forall r \in O^R$ is not acting in a similar manner and output vector $\boldsymbol{y}_o$ is partially expanded to $\left(\boldsymbol{y}_{ro}, \frac{1}{\theta^*}\boldsymbol{y}_{r'o}\right)$ where $r \in O^A$ and $r' \in O^R$. Put differently, an optimal solution for model (1) does not provide an optimal solution to the original DEA problem with ratio data, and vice versa.

Analogously, model (2), in contrast to standard output-oriented DEA model, partially reduces the input vector $\boldsymbol{x}_o$ to $\left(\boldsymbol{x}_{io}, \frac{1}{\phi^*}\boldsymbol{x}_{i'o}\right)$ where $i \in I^A$ and $i' \in I^R$ and it bespeaks that model (1) is unable to result in an optimal solution which corresponds to an optimal solution calculated from the original DEA problem with ratio data.

Flaw 2. Unlike the standard VRS models, we point out a DMU is not necessarily *BCC-efficient* by employing models (1) and (2) with $\vartheta(\gamma) = \vartheta(VRS)$ even if a minimum ratio input value between every ratio input or a maximum ratio output value between every ratio output.

Flaw 3. Let $\vartheta(\gamma) = \vartheta(VRS)$, $\underline{y}_{rp} = \min\left\{\underline{y}_{rj}: j = 1, \dots, n\right\}$ for $r \in O$, and $\bar{x}_{iq} = \max\left\{\bar{x}_{ij}: j = 1, \dots, n\right\}$ for $i \in I$. DMU_p and DMU_q are identified as *BCC-efficient* in models (1) and (2) (see Cooper et al. 2007; p. 93).

2.2. Solution 2 of EA

EA states that solution 1 may not be appropriate to differentiate among efficient units, especially when one evaluates the performance of a small number of DMUs against many ratios and absolute variables. EA, therefore, presents an alternative solution, so-called *solution 2*. Given that some inputs and/or outputs in the form of ratios may violate the typical convexity axioms in DEA, EA revisits this underlying axiom by defining the alternative convex combination for ratio measures. Their general input- and output-oriented models in the presence of input and output ratios are expressed as follows:

IN-Sol.2 model

$$\begin{aligned}
&\min \theta \\
&\text{s.t.} \\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io} && i \in I^A \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} && r \in O^A \\
&\textstyle\sum_{j=1}^{n} \lambda_j \bar{x}_{ij} \le \theta x_{io} \sum_{j=1}^{n} \lambda_j \underline{x}_{ij} && i \in I^R \\
&\textstyle\sum_{j=1}^{n} \lambda_j \left(\bar{y}_{rj} - y_{ro}\underline{y}_{rj}\right) \ge 0 && r \in O^R \\
&\lambda_j \in \vartheta(\gamma)
\end{aligned} \tag{3}$$

OUT-Sol.2 model

$$\begin{aligned}
&\max \phi \\
&\text{s.t.} \\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io} && i \in I^A \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \ge \phi y_{ro} && r \in O^A \\
&\textstyle\sum_{j=1}^{n} \lambda_j \left(\bar{x}_{ij} - x_{io}\underline{x}_{ij}\right) \le 0 && i \in I^R \\
&\textstyle\sum_{j=1}^{n} \lambda_j \bar{y}_{rj} \ge \phi y_{ro} \sum_{j=1}^{n} \lambda_j \underline{y}_{rj} && r \in O^R \\
&\lambda_j \in \vartheta(\gamma)
\end{aligned} \tag{4}$$

Models (3) and (4) are nonlinear programming models due to the multiplier terms $\theta\lambda_j$ and $\phi\lambda_j$. However, when $I^R = \emptyset$ in model (3) and $O^R = \emptyset$ in model (4), the models turn into to be linear. EA lays the emphasis on the VRS technology ($\vartheta(VRS)$) for input- and output-oriented for three cases; (1) the problem includes the ratio inputs, absolute inputs and absolute outputs, i.e., $\{I^R$, I^A and $O^A\}$, (2) the problem consists of the ratio outputs, absolute inputs and absolute outputs, i.e., $\{O^R$, I^A and $O^A\}$, and (3) the problem entails the ratio inputs, ratio outputs, absolute inputs and absolute outputs $\{I^R$, O^R, I^A and $O^A\}$. In total, EA presents six VRS models to cover various situations where two of their models are solely linear.

Remark 1. The production possibility set (PPS) of models (3) and (4) can be defined as follows:

$$P(\vartheta(\gamma)) = \left\{ (\boldsymbol{x}, \boldsymbol{y}) \middle| \begin{array}{l} x_i \ge \sum_{j=1}^{n} \lambda_j x_{ij} (\forall i \in I^A), x_i \ge \frac{\sum_{j=1}^{n} \lambda_j \bar{x}_{ij}}{\sum_{j=1}^{n} \lambda_j \underline{x}_{ij}} (\forall i \in I^R), \\ y_r \le \sum_{j=1}^{n} \lambda_j y_{rj} \ (\forall r \in O^A), y_r \le \frac{\sum_{j=1}^{n} \lambda_j \bar{y}_{rj}}{\sum_{j=1}^{n} \lambda_j \underline{y}_{rj}} \ (\forall\, r \in O^R), \\ \lambda_j \in \vartheta(\gamma) \ (\forall j) \end{array} \right\} \subset \mathbb{R}^{m+s}$$

Remark 2. The IN-Sol.2 model (3) treats the output ratio $\boldsymbol{y}_r = (y_{r1}, \dots, y_{rn})$ for $r \in O^R$ as an absolute output $\begin{pmatrix} \bar{y}_{r1} \\ \vdots \\ \bar{y}_{rn} \end{pmatrix} - y_{ro} \begin{pmatrix} \underline{y}_{r1} \\ \vdots \\ \underline{y}_{rn} \end{pmatrix}$ where its o^{th} component becomes always zero, that is, the absolute output vector for DMU_o is the zero vector $\mathbf{0}_{|O^R|}$[1]. Analogously, in the OUT-Sol.2 model (4), the input ratio $\boldsymbol{x}_i = (x_{i1}, \dots, y_{mn})$ for $i \in I^R$ is considered as an absolute input $\begin{pmatrix} \bar{x}_{1j} \\ \vdots \\ \bar{x}_{mj} \end{pmatrix} - x_{io} \begin{pmatrix} \underline{x}_{1j} \\ \vdots \\ \underline{x}_{mj} \end{pmatrix}$ where the absolute input vector for DMU_o is $\mathbf{0}_{|I^R|}$.

Referring to Cooper et al. (2007; p. 93), a DMU with a minimum input value among any input, or a maximum output value among any output is *BCC-efficient*. However, the following lemmas show that a given DMU with a maximum output ratio or a minimum input ratio value, regardless of the assumed technology, is *efficient.*

Lemma 1. A DMU_p that has a maximum output ratio is *efficient* in model (3).

Proof. Without the loss of generality, let $\frac{\bar{y}_{rp}}{\underline{y}_{rp}} > \frac{\bar{y}_{rj}}{\underline{y}_{rj}}$ $(\forall j \neq p)$. Apropos of **Remark 1**, $\bar{y}_{rj} - y_{rp}\underline{y}_{rj} < 0; \forall j \neq p$ and the constraint $\sum_{j=1}^{n} \lambda_j \left(\bar{y}_{rj} - y_{rp}\underline{y}_{rj} \right) \geq 0$ of model (3) has the unique solution $(\theta^*, \boldsymbol{\lambda}^*) = (1, \boldsymbol{e}_p)$ where $\boldsymbol{e}_p$ is the p^{th} unit vector[2] which implies that DMU_p is efficient. ∎

Lemma 2. A DMU_q that has a minimum input ratio is *efficient* in model (4).

Proof. There is no loss of generality in assuming $\frac{\bar{x}_{iq}}{\underline{x}_{iq}} < \frac{\bar{x}_{ij}}{\underline{x}_{ij}}$ $(\forall j \neq q)$. Then, $\bar{x}_{ij} - x_{ip}\underline{x}_{ij} > 0$ $(\forall j \neq q)$ apropos of **Remark 1** and the constraint $\sum_{j=1}^{n} \lambda_j \left(\underline{x}_{ij} - x_{iq}\bar{x}_{ij} \right) \leq 0$ of model (4) has the unique solution $(\phi^*, \boldsymbol{\lambda}^*) = (1, \boldsymbol{e}_q)$ which implies that DMU_q is efficient. ∎

Let us look at the following flaws and improper characteristics of the EA models:

Flaw 4. Let $S_1(\vartheta(\gamma))$ and $S_2(\vartheta(\gamma))$ be the feasible region of models (1) and (3), respectively. Although EA (Theorem 3, p. 491) tries to prove that $S_1(\vartheta(VRS)) \subseteq S_2(\vartheta(VRS))$ in the absence of ratio inputs, i.e., $I^R = \emptyset$, it does not succeed. Consider the following counterexample involving two DMUs with single absolute input x, single absolute output y_1, and single ratio output $y_2 = \frac{\bar{y}_2}{\underline{y}_2}$:

Table 1. A counterexample

DMUs	x	y_1	$\bar{y}_2$	$\underline{y}_2$
DMU_1	1	2	3	2
DMU_2	2	3	5	4

[1] $\mathbf{0}_{|O^R|}$ stands for the origin in $\mathbb{R}^{|O^R|}$ space, i.e. $\mathbf{0}_{|O^R|} = (0, \dots, 0) \in \mathbb{R}^{|O^R|}$.

[2] p^{th} unit vector is a vector having a zero components, except for a 1 in the P^{th} position (see Bazaraa, Jarvis, & Sherali (2010) p. 46)

It is straightforward to verify that $(\lambda_1, \lambda_2, \theta) = (0,1,2) \in S_1(\vartheta(VRS))$ when DMU_1 is evaluated but $(0,1,2) \notin S_2(\vartheta(VRS))$[3]. On the other hand, since $S_p(\vartheta(VRS))$ for $p = 1,2$ is a subset of $S_p(\vartheta(CRS))$, $S_p(\vartheta(IRS))$ and $S_p(\vartheta(DRS))$, we can conclude in general that $S_1(\vartheta(\gamma)) \not\subseteq S_2(\vartheta(\gamma))$. Analogously, let $D_1(\vartheta(\gamma))$ and $D_2(\vartheta(\gamma))$ be the feasible region of models (2) and (4), respectively. It is obvious that $D_1(\vartheta(\gamma)) \not\subseteq D_2(\vartheta(\gamma))$, regardless of the assumed technology.

Flaw 5. Though the proposed models of EA are developed based on the VRS assumption, the authors (p. 495) claims that their proposed models can be extended to other technologies such as CRS, IRS and DRS. However, we here show that apart from the VRS and IRS models this claim apropos of Solution 2 is no longer valid under some conditions.

Let us disregard absolute outputs in the IN-Sol.2 model (3), i.e. $O^A = \emptyset$, and absolute inputs in the OUT-Sol.2 model (4), i.e. $I^A = \emptyset$. Consequently, we arrive at the following nonlinear models without absolute outputs and absolute inputs in models (3) and (4), respectively:

$$\begin{aligned} &\min \theta \\ &\text{s.t.} \\ &\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io} && i \in I^A \\ &\textstyle\sum_{j=1}^{n} \lambda_j \bar{x}_{ij} \le \theta x_{io} \sum_{j=1}^{n} \lambda_j \underline{x}_{ij} && i \in I^R \\ &\textstyle\sum_{j=1}^{n} \lambda_j \left(\bar{y}_{rj} - y_{ro}\underline{y}_{rj}\right) \ge 0 && r \in O^R \\ &\lambda_j \in \vartheta(\gamma) \end{aligned} \tag{5}$$

$$\begin{aligned} &\max \phi \\ &\text{s.t.} \\ &\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \ge \phi y_{ro} && r \in O^A \\ &\textstyle\sum_{j=1}^{n} \lambda_j \left(\bar{x}_{ij} - x_{io}\underline{x}_{ij}\right) \le 0 && i \in I^R \\ &\textstyle\sum_{j=1}^{n} \lambda_j \bar{y}_{rj} \ge \phi y_{ro} \sum_{j=1}^{n} \lambda_j \underline{y}_{rj} && r \in O^R \\ &\lambda_j \in \vartheta(\gamma) \end{aligned} \tag{6}$$

Consequently, the following lemmas prove that models (5) and (6) are not capable of evaluating the DMUs for $\vartheta(CRS)$ and $\vartheta(DRS)$ because the optimum solution of the former model is always equal to zero and there is no optimal solution for the latter model.

Lemma 3. The optimal objective value of (5) is zero under the CRS and DRS assumptions.

Proof. Let $\tilde{\lambda}_j = 0; \forall j$ and $\tilde{\theta} = 0$. An easy computation shows that the vector $(\tilde{\boldsymbol{\lambda}}, \tilde{\theta}) = (\mathbf{0}_n, 0)$ is a feasible solution of model (5) which is optimum as well. ■

Lemma 4. The optimal objective value of model (6) is unbounded under the CRS and DRS assumptions.

[3] It should be noted that a feasible (non-optimal) solution is here presumed and this is due to $\theta = 2$ in the input-oriented models.

Proof. Let $\tilde{\lambda}_j = 0; \forall j \neq o$ and $\tilde{\lambda}_o \to +\infty$. The vector $(\tilde{\boldsymbol{\lambda}}, \tilde{\theta})$ is a feasible solution of model (6) for any $\tilde{\theta} > 0$. Therefore, the objective value can be driven to $+\infty$, or it is unbounded. ■

Flaw 6. The CRS model exhibits that the optimal objective value of the output-oriented model θ links to that of the inverse of the optimal objective value $1/\theta$ obtained from the input-oriented model. However, the optimal objective value of IN-Sol.2 model (3) under the CRS ($\vartheta(CRS)$) does not necessarily relate to the optimal objective value of OUT-Sol.2 model (4). In other words, in contrast to the traditional DEA models, the IN-Sol.2 model (3) is not equivalent to the OUT-Sol.2 model (4) when $\vartheta(\gamma) = \vartheta(CRS)$. We will prove that our improved models keep this important feature.

It is ultimately worth noting that the nonlinear IN-Sol.2 model (3) can be transformed to the linear programming model if there is no ratio input, i.e., $I^R = \emptyset$; however, the OUT-Sol.2 model (4) is still nonlinear as follows:

$$
\begin{array}{ll}
\max \phi & \\
\text{s.t.} & \\
\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} & i \in I^A \\
\sum_{j=1}^{n} \lambda_j y_{rj} \geq \phi y_{ro} & r \in O^A \\
\sum_{j=1}^{n} \lambda_j \underline{y}_{rj} \geq \phi y_{ro} \sum_{j=1}^{n} \lambda_j \underline{y}_{rj} & r \in O^R \\
\lambda_j \in \vartheta(\gamma) &
\end{array}
\qquad (7)
$$

The same issue exists for the OUT-Sol.2 model (4) in the absence of ratio output.

3. Proposed solutions

In this section, we provide a sort of remedies to treat the flaws of the EA models listed and explained in the preceding section.

3.1. Remedy for Solution 1

In order to tackle the problems argued in Subsection 2.1, we propose the following input- and output-oriented models in which ratio measures are deemed to be non-discretionary variables:

<u>*Modified IN-Sol.1 model*</u>

$$
\begin{array}{ll}
\min \theta & \\
\text{s.t.} & \\
\sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io} & i \in I^A \\
\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} & r \in O^A \\
\sum_{j=1}^{n} \lambda_j \bar{x}_{ij} \leq \bar{x}_{io} & i \in I^R \\
\sum_{j=1}^{n} \lambda_j \underline{x}_{ij} \geq \underline{x}_{io} & i \in I^R \\
\sum_{j=1}^{n} \lambda_j \underline{y}_{rj} \leq \underline{y}_{ro} & r \in O^R \\
\sum_{j=1}^{n} \lambda_j \bar{y}_{rj} \geq \bar{y}_{ro} & r \in O^R \\
\lambda_j \in \vartheta(\gamma) &
\end{array}
\qquad (8)
$$

<u>*Modified OUT-Sol.1 model*</u>

$$
\begin{array}{ll}
\max \phi & \\
\text{s.t.} & \\
\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} & i \in I^A \\
\sum_{j=1}^{n} \lambda_j y_{rj} \geq \phi y_{ro} & r \in O^A \\
\sum_{j=1}^{n} \lambda_j \bar{x}_{ij} \leq \bar{x}_{io} & i \in I^R \\
\sum_{j=1}^{n} \lambda_j \underline{x}_{ij} \geq \underline{x}_{io} & i \in I^R \\
\sum_{j=1}^{n} \lambda_j \underline{y}_{rj} \leq \underline{y}_{ro} & r \in O^R \\
\sum_{j=1}^{n} \lambda_j \bar{y}_{rj} \geq \bar{y}_{ro} & r \in O^R \\
\lambda_j \in \vartheta(\gamma) &
\end{array}
\qquad (9)
$$

The modified input-oriented model (8) reduces the input vector $\boldsymbol{x}_o = (\boldsymbol{x}_{io}, \boldsymbol{x}_{i'o})$, $i \in I^A, i' \in I^R$ to $(\theta^* \boldsymbol{x}_{io}, \boldsymbol{x}_{i'o})$ where the output vector $\boldsymbol{y}_o$ is fixed. Analogously, in the modified output-oriented model (9) the output vector $\boldsymbol{y}_o = (\boldsymbol{y}_{ro}, \boldsymbol{y}_{r'o})$, $r \in O^A$, $r' \in O^R$ is increased to $\boldsymbol{y}_o = (\phi^* \boldsymbol{y}_{ro}, \boldsymbol{y}_{r'o})$ while the input vector $\boldsymbol{x}_o$ is kept unchanged. It is conspicuous that at least one absolute input and one absolute output are the essence of models (8) and (9), respectively. If one is only searching for the relative efficiency and the model orientation does not play a part in the evaluation process, models (9) and (8) can be utilized in the case of no absolute input and no absolute output, respectively.

Models (8) and (9) therefore enable us to satisfy the proportionality assumption of standard DEA models. It is worthwhile to spell out that these models are no longer valid when considering ratio measures as discretionary variables is indispensable.

The following lemmas show the relationship between the optimal solutions of the above models and the original DEA models.

Lemma 5. The optimal objective value of model (8) is a lower bound for the optimal objective value of model (1).

Proof. Let $(\theta^*, \boldsymbol{\lambda}^*)$ and $(\bar{\theta}^*, \bar{\boldsymbol{\lambda}}^*)$ be the optimal solution of models (1) and (8), respectively. It is plain to verify that $(\theta^*, \boldsymbol{\lambda}^*)$ is a feasible solution of model (8) and hence $\bar{\theta}^* \leq \theta^*$. ■

Lemma 6. The optimal objective value of model (9) is an upper bound for the optimal objective value of model (2).

Proof. The proof is similar to that of Lemma 5 (omitted).

3.2. Remedy for Solution 2

With the aim of coping with Flaws 5 and 6, we first improve the nonlinear IN-Sol.2 model. To do this, we [temporarily] assume $I^R = \emptyset$ and $\vartheta(\gamma) = \vartheta(CCR)$ to utilize an interesting property in which the input-and output-oriented DEA models are equivalent. However, these assumptions will be relaxed later.

Consider the primal envelopment IN.Sol.2 model and its dual multiplier formulation without ratio input under the CRS assumption:

Primal IN.Sol.2 model

$$
\begin{aligned}
&\min \theta \\
&\text{s.t.} \\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io} && i \in I^A \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro} && r \in O^A \qquad (10) \\
&\textstyle\sum_{j=1}^{n} \lambda_j \left(\bar{y}_{rj} - y_{ro}\underline{y}_{rj}\right) \ge 0 && r \in O^R \\
&\lambda_j \ge 0
\end{aligned}
$$

Dual IN.Sol.2 model

$$
\begin{aligned}
&\max \textstyle\sum_{r\in O^A} u_r y_{ro} \\
&\text{s.t.} \\
&\textstyle\sum_{i\in I^A} v_i x_{io} = 1 \qquad (11) \\
&\textstyle\sum_{r\in O^A} u_r y_{rj} + \sum_{r\in O^R} u_r \left(\bar{y}_{rj} - y_{ro}.\underline{y}_{rj}\right) - \sum_{i\in I^A} v_i x_{ij} \le 0 && \forall j \\
&u_r, v_i \ge 0 && \forall r, i
\end{aligned}
$$

Under the CRS assumption, there is a direct link between an optimal solution of the input- and output-oriented models (see Cooper, Seiford & Tone, 2007, p.59). As a result, the input-oriented [dual] multiplier model can be adapted to formulate the output-oriented [dual] multiplier model (12). Model (12) and its dual are expressed below:

Dual OUT.Sol.2 model

$$
\begin{aligned}
&\min \textstyle\sum_{i\in I^A} v_i x_{io} \\
&\text{s.t.} \\
&\textstyle\sum_{r\in O^A} u_r y_{ro} = 1 \qquad (12) \\
&\textstyle\sum_{r\in O^A} u_r y_{rj} + \sum_{r\in O^R} u_r \left(\bar{y}_{rj} - y_{ro}.\underline{y}_{rj}\right) - \sum_{i\in I^A} v_i x_{ij} \le 0 && \forall j \\
&u_r, v_i \ge 0 && \forall r, i
\end{aligned}
$$

Primal OUT.Sol.2 model

$$
\begin{aligned}
&\max \phi \\
&\text{s.t.} \\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io} && i \in I^A \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \ge \phi y_{ro} && r \in O^A \qquad (13) \\
&\textstyle\sum_{j=1}^{n} \lambda_j \left(\bar{y}_{rj} - y_{ro}\underline{y}_{rj}\right) \ge 0 && r \in O^R \\
&\lambda_j \ge 0
\end{aligned}
$$

Model (13) is called the primal envelopment OUT.Sol.2 model. At present, let us return to the OUT.Sol.2 model with $I^R = \emptyset$ and $\vartheta(\gamma) = \vartheta(CCR)$ proposed by EA as formulated below:

$$\begin{aligned}
&\max \phi \\
&\text{s.t.} \\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} && i \in I^A \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \geq \phi y_{ro} && r \in O^A \\
&\textstyle\sum_{j=1}^{n} \lambda_j \left(\bar{y}_{rj} - \phi y_{ro}.\underline{y}_{rj}\right) \geq 0 && r \in O^R \\
&\lambda_j \geq 0
\end{aligned} \tag{14}$$

The major difference between models (13) and (14) is the set of output-ratio constraints that includes the source of nonlinearity in (14). One instant thrust to overcome this inconsistency is to classify the ratio output into a non-discretionary factor as appropriate. It should be emphasized that the above argument can be extended to the OUT-Sol.2 model with $O^R = \emptyset$ and $\vartheta(\gamma) = \vartheta(CCR)$. However, it should not be extendable and applicable to all situations and it sheds light on future research directions.

It is now supposed that the problem includes at least one absolute input for an input-orientation and one absolute output for an output orientation. We generalize our idea and propose the following pair of input- and output-oriented [envelopment] models to deal with ratio factors, regardless of the assumed technology:

Improved IN-Sol.2 model

$$\begin{aligned}
&\min \theta \\
&\text{s.t.} \\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io} && i \in I^A \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} && r \in O^A \\
&\textstyle\sum_{j=1}^{n} \lambda_j \left(\bar{x}_{ij} - x_{io}\underline{x}_{ij}\right) \leq 0 && i \in I^R(ND) \\
&\textstyle\sum_{j=1}^{n} \lambda_j \left(\bar{y}_{rj} - y_{ro}\underline{y}_{rj}\right) \geq 0 && r \in O^R(ND) \\
&\lambda_j \in \vartheta(\gamma)
\end{aligned} \tag{15}$$

Improved OUT-Sol.2 model

$$\begin{aligned}
&\max \phi \\
&\text{s.t.} \\
&\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} && i \in I^A \\
&\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \geq \phi y_{ro} && r \in O^A \\
&\textstyle\sum_{j=1}^{n} \lambda_j \left(\bar{x}_{ij} - x_{io}\underline{x}_{ij}\right) \leq 0 && i \in I^R(ND) \\
&\textstyle\sum_{j=1}^{n} \lambda_j \left(\bar{y}_{rj} - y_{ro}\underline{y}_{rj}\right) \geq 0 && r \in O^R(ND) \\
&\lambda_j \in \vartheta(\gamma)
\end{aligned} \tag{16}$$

In the above-modified models, we assume that all input and output ratios are considered as non-discretionary variables and the symbol "ND" refers to these variables. The following theorem proves that contrary to the IN-Sol2 and Out-Sol2 models, an optimal solution of model (15) is related to that of model (16) under CRS assumption:

Theorem 1. Models (15) and (16) are equivalent under CRS assumption.

Proof. Let an optimal solution of model (15) be $(\theta^*, \boldsymbol{\lambda}^*)$. It is plain to verify that $(\frac{1}{\theta^*}, \frac{\boldsymbol{\lambda}^*}{\theta^*})$ is a feasible solution for model (16). Suppose, contrary to our claim, that $(\phi^*, \bar{\boldsymbol{\lambda}}^*)$ is the optimal solution for model (16) and $\phi^* > \frac{1}{\theta^*}$. Then $(\frac{1}{\phi^*}, \bar{\boldsymbol{\lambda}}^*)$ is a feasible solution for model (15) which its objective function value is strictly smaller than the optimal objective value but this contradicts the optimality of (θ^*, λ^*). The reverse can be analogously proved. ■

The following lemma is an important consequence of **Theorem 1**:

Lemma 7. $\theta^* = 1$ if and only if $\phi^* = 1$ when $\vartheta(\gamma) = \vartheta(CRS)$.

It should be underlined here that the feasible region of modified IN- (OUT-) Sol.1 model is not a subset of the feasible region of improved IN- (OUT-) Sol.2 model. Hence, the suggested remedy cannot deal with **Flaw 4**. On the other hand, the PPS of the improved IN- (OUT-) Sol.2 model is not identical and hence the proposed models suffer the multi-PPS issue. Another limitation of the proposed models is that the problem in Lemma 3 is still available, that is, the above models under the CRS or DRS assumption are not capable of assessing the efficiency of DMUs in the situations where we do not have any absolute outputs in model (15) and absolute inputs in model (16).

Interestingly, the pair of multiplier models are formulated for both input- and output orientations as follows:

Improved dual IN-Sol.2 model

$$\max \sum_{r\in O^A} u_r y_{ro} + w_0$$

$$\text{s.t.}$$

$$\sum_{i\in I^A} v_i x_{io} = 1$$

$$\sum_{r\in O^A} u_r y_{rj} + \sum_{r\in O^R} u_r \left(\bar{y}_{rj} - y_{ro}\underline{y}_{rj}\right) + w_0 - \sum_{i\in I^A} v_i x_{ij} - \sum_{i\in I^R} v_i\left(\underline{x}_{ij} - x_{io}\bar{x}_{ij}\right) \le 0, \quad \forall j \qquad (17)$$

$$u_r, v_i \ge 0, \quad \forall r, i$$

$$w_0 \in \mu(RTS)$$

Improved dual OUT-Sol.2 model

$$
\begin{aligned}
&\min \textstyle\sum_{i\in I^A} v_i x_{io} + w_0 \\
&\text{s.t.} \\
&\textstyle\sum_{r\in O^A} u_r y_{ro} = 1 \\
&\textstyle\sum_{r\in O^A} u_r y_{rj} + \sum_{r\in O^R} u_r \left(\bar{y}_{rj} - y_{ro}.\underline{y}_{rj}\right) - \\
&\quad \textstyle\sum_{i\in I^A} v_i x_{ij} - \sum_{i\in I^R} v_i \left(\underline{x}_{ij} - x_{io}\bar{x}_{ij}\right) - w_0 \le 0, \ \forall j \\
&u_r, v_i \ge 0, \quad \forall r, i \\
&w_0 \in \mu(RTS)
\end{aligned} \tag{18}
$$

where $\mu(CRS) = \{w_0 | w_0 = 0\}, \mu(VRS) = \{w_0 | w_0 \in \mathcal{R}\}, \mu(IRS) = \{w_0 | w_0 \in \mathbb{R}^+\}$, and $\mu(DRS) = \{w_0 | w_0 \in \mathbb{R}^-\}$.

The above DEA models are capable of measuring the performance of DMUs by the ratio and absolute factors. In such cases, both the numerator and denominator of a ratio are numeric measures (the set of positive real numbers) such as *Return on Equity* that is calculated as the ratio of *net income* to *equity* in which both the numerator and denominator belong to $\mathbb{R}^+$. However, many real-life applications include an indicator where the numerator and denominator are in a ratio form, and we call it the "*ratio of ratios*". For example, the *savings ratio* is defined as the *average household savings* divided by the *average household disposable income* in which both the numerator and denominator are represented by ratios. Let us now discuss how one adapts the standard (radial) DEA models where there are *ratio of ratios* measures. Let $I^{RR} \subseteq I$ and $O^{RR} \subseteq O$ stand for the sub-index representing "ratio of ratios" inputs and "ratio of ratios" outputs, respectively. We also assume that $\bar{x}_{ij} = \frac{\bar{x}'_{ij}}{\bar{x}''_{ij}}, \forall i \in I^{RR} \left(\bar{y}_{rj} = \frac{\bar{y}'_{rj}}{\bar{y}''_{rj}}, \forall r \in O^{RR}\right)$ and $\underline{x}_{ij} = \frac{\underline{x}'_{ij}}{\underline{x}''_{ij}}, \forall i \in I^{RR} \left(\underline{y}_{rj} = \frac{\underline{y}'_{rj}}{\underline{y}''_{rj}}, \forall r \in O^{RR}\right)$. Consequently, $x_{ij}(y_{rj})$ is expressed as the ratio of ratios $\frac{\bar{x}'_{ij}/\bar{x}''_{ij}}{\underline{x}'_{ij}/\underline{x}''_{ij}} \left(\frac{\bar{y}'_{rj}/\bar{y}''_{rj}}{\underline{y}'_{rj}/\underline{y}''_{rj}}\right)$ or equivalently the ratio of $\frac{\bar{x}'_{ij}\underline{x}''_{ij}}{\underline{x}'_{ij}\bar{x}''_{ij}} \left(\frac{\bar{y}'_{rj}\underline{y}''_{rj}}{\underline{y}'_{rj}\bar{y}''_{rj}}\right)$. To incorporate the above sub-sets into our proposed models, one is in need of adding the following two sets of constraints to the input- and output-oriented [envelopment] models (15) and (16); $\sum_{j=1}^{n} \lambda_j \left(\bar{x}'_{ij}\underline{x}''_{ij} - x_{io}\underline{x}'_{ij}\bar{x}''_{ij}\right) \le 0, \forall i \in I^{RR}$ and $\sum_{j=1}^{n} \lambda_j \left(\bar{y}'_{rj}\underline{y}''_{rj} - y_{ro}\underline{y}'_{rj}\bar{y}''_{rj}\right) \ge 0, \forall r \in O^{RR}$, and regarding the input- and output-oriented [multiplier] models (17) and (18) one requires adding the following component, $\sum_{r\in O^{RR}} u_r \left(\bar{y}'_{rj}\underline{y}''_{rj} - y_{ro}\underline{y}'_{rj}\bar{y}''_{rj}\right)$

and subtracting $\sum_{i\in I^{RR}} v_i\left(\bar{x}'_{ij}\underline{x}''_{ij} - x_{io}\underline{x}'_{ij}\bar{x}''_{ij}\right)$ to and from the second constraint, i.e., $\sum_{r\in O^A} u_r y_{rj} + \sum_{r\in O^R} u_r\left(\bar{y}_{rj} - y_{ro}\underline{y}_{rj}\right) + \sum_{r\in O^{RR}} u_r\left(\bar{y}'_{rj}\underline{y}''_{rj} - y_{ro}\underline{y}'_{rj}\bar{y}''_{rj}\right) + w_0 - \sum_{i\in I^A} v_i x_{ij} - \sum_{i\in I^R} v_i\left(\underline{x}_{ij} - x_{io}\bar{x}_{ij}\right) - \sum_{i\in I^{RR}} v_i\left(\bar{x}'_{ij}\underline{x}''_{ij} - x_{io}\underline{x}'_{ij}\bar{x}''_{ij}\right) \leq 0, \forall j.$

4. Efficiency assessment of universities

This section provides a simple case study to illustrate the shortcomings of the EA approach and the advantages of our developed models. We consider a dataset involving 20 universities with two inputs and two outputs shown in Table 2. In this problem, x_1 and x_2 represent "*percentage of full-time faculty members*" and "*total cost*" which are ratio (i.e., $\bar{x}_1/\underline{x}_1$) and absolute factors, respectively, and y_1 and y_2 represent "*research income in million £*" and "*percentage of degree awarded*" which are absolute and ratio (i.e., $\bar{y}_2/\underline{y}_2$) factors, respectively.

Table 2. The input and output values for 20 universities

University	Input-Ratio			x_2	y_1	Output-Ratio		
	$\underline{x}_1$	$\bar{x}_1$	x_1			$\underline{y}_2$	$\bar{y}_2$	y_2
U1	171	130	0.76	165	30	6500	1300	0.20
U2	220	150	0.68	200	40	8000	2000	0.25
U3	70	50	0.71	80	12	2000	300	0.15
U4	110	87	0.79	120	30	4000	720	0.18
U5	139	130	0.93	150	30	6000	600	0.10
U6	250	205	0.82	210	45	10000	2500	0.25
U7	268	250	0.93	330	30	5000	500	0.10
U8	277	255	0.92	310	40	4909	540	0.11
U9	298	245	0.82	300	60	10000	1500	0.15
U10	59	40	0.67	90	12	2000	320	0.16
U11	112	100	0.89	190	20	6666	1000	0.15
U12	191	140	0.73	180	30	14000	1400	0.10
U13	141	129	0.91	150	12	9090	1000	0.11
U14	123	110	0.89	130	10	3000	360	0.12
U15	156	130	0.83	190	35	4500	450	0.10
U16	156	155	0.99	155	20	10000	1000	0.10
U17	162	135	0.83	185	25	3545	390	0.11
U18	216	134	0.62	335	30	2241	650	0.29
U19	211	133	0.63	300	35	2000	700	0.35
U20	555	500	0.90	190	12	2909	320	0.11

Let us first solve the conventional input- and output-oriented CCR and BCC models with x_1, x_2, y_1 and y_2. The 2nd column of Table 3 shows the results associated to the input- and output-oriented CCR model, and the 3rd and 4th columns of Table 3 report the results associated to the input- and output-oriented BCC model, respectively. However, due to ratio measures, these models are not able to correctly measure the efficiency of universities. We thereby utilize models (1) and (2) as Solution 1 of EA under the CRS and VRS assumptions as presented in the last three columns of Table 3.

Table 3. The efficiency scores of solution 1 for different cases.

University	Efficiency score					
	CCR model	Input-oriented BCC model	Output-oriented BCC model	Sol.1 model $\vartheta(CRS)$	IN-Sol.1 model $\vartheta(VRS)$	OUT-Sol.1 model $\vartheta(VRS)$
U_1	0.8866	0.9217	0.9044	0.9141	0.9371	0.9297
U_2	1	1	1	1	1	1
U_3	1	1	1	0.9401	1	1
U_4	1	1	1	1	1	1
U_5	0.8163	0.8341	0.8571	0.8623	0.8627	0.8681
U_6	1	1	1	1	1	1
U_7	0.4628	0.6973	0.5397	0.7815	0.7835	0.8471
U_8	0.6355	0.7355	0.6825	0.8571	0.9266	0.9769
U_9	1	1	1	0.9270	1	1
U_{10}	1	1	1	1	1	1
U_{11}	0.5623	0.7468	0.6109	0.7622	0.8452	0.8088
U_{12}	0.7674	0.9236	0.7872	0.9430	0.9525	0.9493
U_{13}	0.4662	0.7318	0.5187	0.7814	0.8439	0.8051
U_{14}	0.5427	0.7511	0.6117	0.8245	0.8482	0.8412
U_{15}	0.8326	0.8686	0.8400	0.9238	0.9639	0.9739
U_{16}	0.5214	0.7112	0.5581	0.7511	0.8251	0.7812
U_{17}	0.6068	0.8087	0.6122	0.8788	0.8877	0.8832
U_{18}	0.8616	1	1	1	1	1
U_{19}	1	1	1	1	1	1
U_{20}	0.4016	0.7333	0.4427	1	1	1

Note that models (1) and (2) take numerators and denominator of ratio measures into account to evaluate the performance of universities. It is clear that models (1) and (2) fails to satisfy the concept of Farrell measure. For example, the efficiency of U_1 for the input-oriented model (1) under the CRS is 0.9141, meaning that all the inputs of U_1 are proportionally reduced by 0.9141 without changing the amount of its outputs. That is, considering the denominator of output ratio ($\underline{y}_2 = 6500$) as an input in model (1) leads to changing the value of $\underline{y}_2$ to 5941.65 and consequently increasing in the corresponding output $\frac{\bar{y}_2}{\underline{y}_2} = 0.219$ which contradicts the concept of Farrell measure that says the outputs are fixed. Similarly, the output-oriented model (2) suffers from this fundamental shortcoming. Given that $\max\{\bar{x}_{1,j}: j = 1, \dots, 20\} = \bar{x}_{1,20} = 500$ and $\min\{\underline{y}_{2,j}: j = 1, \dots, 20\} = \underline{y}_{2,3} = \underline{y}_{2,10} =$

$\underline{y}_{2,19} = 2000$, U_{20} is *BCC-efficient* in model (1) under the VRS and U_3, U_{10} and U_{20} are *BCC-efficient* in terms of model (2) under the VRS. In comparison with conventional BCC models (both orientations) U_{20} is classified in the set of BCC-inefficient units. To deal with these inevitable problems of Solution 1 of EA, we introduce the input- and output-oriented models (7) and (8) where the ratio factors are assumed to be non-discretionary variables. The results computed from models (7) and (8) under the CRS and VRS assumptions are listed in Table 4. As can be seen, in line with our expectation, the efficiency calculated under CRS does not exceed that calculated under VRS. It also is observed that the efficiency of each university calculated from (7) and (8) is not greater than the corresponding efficiency calculated from (1) and (2) as input and/or output ratio factors are considered as non-discretionary variables.

Table 4. The efficiency scores of improved solution 1 for different cases.

University	Efficiency scores			
	Modified IN-Sol.1 model $\vartheta(CRS)$	Modified OUT-Sol.1 model $\vartheta(CRS)$	Modified IN-Sol.1 model $\vartheta(VRS)$	Modified OUT-Sol.1 model $\vartheta(VRS)$
U_1	0.8784	0.8005	0.9146	0.8201
U_2	1	1	1	1
U_3	0.8300	0.7965	1	1
U_4	1	1	1	1
U_5	0.8549	0.8042	0.8574	0.8571
U_6	1	1	1	1
U_7	0.4971	0.5784	0.5123	0.7066
U_8	0.7493	0.8205	0.8818	0.9650
U_9	0.8740	0.8240	1	1
U_{10}	1	1	1	1
U_{11}	0.4904	0.6286	0.6388	0.6297
U_{12}	0.8625	0.7672	0.8982	0.7843
U_{13}	0.6467	0.3252	0.8304	0.3429
U_{14}	0.4973	0.3975	0.7262	0.4252
U_{15}	0.8378	0.9130	0.9300	0.9739
U_{16}	0.6802	0.5368	0.8251	0.5581
U_{17}	0.6645	0.7688	0.6759	0.7804
U_{18}	1	1	1	1
U_{19}	1	1	1	1
U_{20}	1	1	1	1

Let us now focus on Solution 2 of EA. We measure the efficiency of each university using the input- and output-oriented models (3) and (4) that are nonlinear at large. We utilize

BARON as a popular solver in general algebraic modelling system[4] (GAMS) to solve these nonlinear problems. Table 5 presents the efficiency scores for twenty universities under the CRS and VRS assumptions by means of models (3) and (4). Note that "lo" and "gl" in the parentheses represent the *local* and *global* optimal solutions. As can be seen, units {2, 4, 6, 18, 19} are efficient in both CRS orientations and units {2, 3, 4, 6, 9, 10, 18, 19} are efficient in both VRS orientations. U_{19} is *efficient* from the input-oriented viewpoint since this unit has a maximum output ratio value, i.e., $\max\{y_{2j}: j = 1, \ldots, 20\} = y_{2,19} = 0.62$ (see Lemma 1), and U_{18} is *efficient* from the output-oriented viewpoint since this unit has a minimum input ratio value, i.e., $\min\{x_{1j}: j = 1, \ldots, 20\} = x_{1,18} = 0.35$ (see Lemma 2).

Table 5. The efficiency scores of solution 2 for different cases.

University	Efficiency score			
	IN-Sol.2 model $\vartheta(CRS)$	OUT-Sol.2 model $\vartheta(CRS)$	IN-Sol.2 model $\vartheta(VRS)$	OUT-Sol.2 model $\vartheta(VRS)$
U_1	0.8998 (*lo*)	0.8403 (*lo*)	0.9162 (*lo*)	0.8538 (*lo*)
U_2	1 (*lo*)	1 (*lo*)	1 (*lo*)	1 (*lo*)
U_3	0.9323 (*lo*)	0.6959 (*gl*)	1 (*lo*)	1 (*lo*)
U_4	1 (*lo*)	1 (*lo*)	1 (*lo*)	1 (*lo*)
U_5	0.8270 (*lo*)	0.8000 (*gl*)	0.8299 (*lo*)	0.8571 (*gl*)
U_6	1 (*lo*)	1 (*lo*)	1 (*lo*)	1 (*lo*)
U_7	0.6911 (*lo*)	0.4134 (*lo*)	0.6932 (*lo*)	0.5455 (*lo*)
U_8	0.7286 (*lo*)	0.5373 (*lo*)	0.7390 (*lo*)	0.6848 (*lo*)
U_9	0.8897 (*lo*)	0.8070 (*lo*)	1 (*lo*)	1 (*lo*)
U_{10}	0.9962 (*lo*)	0.7553 (*gl*)	1 (*lo*)	1 (*lo*)
U_{11}	0.7321 (*lo*)	0.5838 (*lo*)	0.7420 (*lo*)	0.6046 (*lo*)
U_{12}	0.9207 (*lo*)	0.7407 (*gl*)	0.9254 (*lo*)	0.7827 (*gl*)
U_{13}	0.6919 (*lo*)	0.4306 (*lo*)	0.7237 (*lo*)	0.4674 (*lo*)
U_{14}	0.7049 (*lo*)	0.4591 (*lo*)	0.7473 (*lo*)	0.5315 (*lo*)
U_{15}	0.8551 (*lo*)	0.7368 (*gl*)	0.8564 (*lo*)	0.8400 (*gl*)
U_{16}	0.6824 (*lo*)	0.5245 (*lo*)	0.7122 (*lo*)	0.5581 (*gl*)
U_{17}	0.8024 (*lo*)	0.5559 (*lo*)	0.8097 (*lo*)	0.6122 (*gl*)
U_{18}	1 (*lo*)	1 (*lo*)	1 (*lo*)	1 (*lo*)
U_{19}	1 (*lo*)	1 (*lo*)	1 (*lo*)	1 (*lo*)
U_{20}	0.6898 (*lo*)	0.4099 (*lo*)	0.7189 (*lo*)	0.4420 (*lo*)

Table 6 shows the efficiency of universities in modified linear models (19) and (20). One of main advantages of our proposed method is the linearity of models (21) and (22) against the nonlinear (3) and (4) that guarantees the global optimal solution. The lack of relationship between the efficiency measures of input- and output-oriented models under the CRS assumption are observable in Solution 2 of EA (see Flaw 6). For example, the efficiency

[4] Free demo version is available at www.gams.com.

score of U_1 is 0.8998 and 0.8403 in the IN-Sol.2 model (3) and OUT-Sol.2 model (4), respectively, under the CRS, which are not plainly identical. Interestingly, this issue is treated by dint of the modified models (23) and (24) as shown in Table 6.

Table 6. A comparison with the EA and our approaches

University	Efficiency score			
	Improved IN-Sol.2 model $\vartheta(CRS)$	Improved OUT-Sol.2 model $\vartheta(CRS)$	Improved IN-Sol.2 model $\vartheta(VRS)$	Improved OUT-Sol.2 model $\vartheta(VRS)$
U_1	0.7650	0.7650	0.7850	0.8187
U_2	1	1	1	1
U_3	0.6959	0.6959	1	1
U_4	1	1	1	1
U_5	0.8000	0.8000	0.8000	0.8571
U_6	1	1	1	1
U_7	0.3636	0.3636	0.3636	0.5000
U_8	0.5161	0.5161	0.5806	0.6667
U_9	0.8000	0.8000	1	1
U_{10}	0.7553	0.7553	1	1
U_{11}	0.4211	0.4211	0.5146	0.4800
U_{12}	0.7407	0.7407	0.7627	0.7827
U_{13}	0.3200	0.3200	0.5333	0.3429
U_{14}	0.3077	0.3077	0.6154	0.3158
U_{15}	0.7368	0.7368	0.7895	0.8400
U_{16}	0.5161	0.5161	0.6308	0.5581
U_{17}	0.5405	0.5405	0.5886	0.6122
U_{18}	1	1	1	1
U_{19}	1	1	1	1
U_{20}	0.2526	0.2526	0.4211	0.2880

If one removes the first output y_1 (*research income in million £*) as an absolute measure from the performance evaluation process, the efficiency of all universities calculated from model (5) under the CRS and DRS leads to zero, that is, Solution 2 of EA fails to evaluate the performance of units in this situation (see Lemma 3 and Flaw 6). Analogously, removing the absolute input x_2 (*total cost*) from the evaluation analysis under the CRS and DRS causes zero value of efficiency for all universities. It should be emphasized that our proposed models are also unable to assess the universities in the aforesaid special cases.

5. Concluding remarks

Ratio measures such as the ratio of government expenditures to GDP are the typical type of indicators in public and private sectors. Although the employment of conventional DEA models often fail the basic axioms such as the convexity, the DEA literature is not rich in tackling ratio measures and Emrouznejad & Amin (2009)'s work that includes two solutions is the most appealing approach among them to treat ratio measure in the standard DEA model.

Emrouznejad & Amin (2009) studied the problem from input and output orientations with emphasis on the VRS technology where the data encompasses output- and/or input- ratio variables. Emrouznejad & Amin (2009) admitted that the first solution has insufficient power to discriminate between DMUs and the second solution requires to think of a nonlinear programming model in many cases. Besides these problems, we show several flaws in both of the solutions which degrade the applicability of the models developed by Emrouznejad & Amin (2009). We provide an auxiliary investigation of Emrouznejad & Amin (2009)'s solutions so as to propose the modified models which enable to handle the problems involved in their models. A numerical illustration is presented to demonstrate minutely the flaws together with the results of the modified models proposed in this paper as alternative ways for DEA efficiency assessment and ranking in the presence of ratio measures.

Future work is needed to develop DEA models for ratio data when the underlying measures of all ratios (the numerator and denominator) are available. Particularly, the efficiency evaluation of those problems that do not include absolute input and absolute output in output- and input-oriented models, respectively, could be an interesting path of further work.

References

Bazaraa, M. S., Jarvis, J. J., & Sherali, H. D. (2010). *Linear programming and network flows* (4th ed.). Hoboken, New Jersey: John Wiley & Sons.

Carayannis, E. G., Grigoroudis, E., & Goletsis, Y. (2016). A multilevel and multistage efficiency evaluation of innovation systems: A multiobjective DEA approach. *Expert Systems with Applications*, *62*, 63–80.

Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research*, *2*, 429–444.

Cooper, W. W., Seiford, L. M., & Tone, K. (2007). *Data envelopment analysis: A comprehensive text with models, applications, references and DEA-solver software* (2nd edi.). Springer US.

Emrouznejad, A., & Amin, G. R. (2009). DEA models for ratio data: Convexity consideration. *Applied Mathematical Modelling*, *33*(1), 486–498.

Gidion, D. K., Hong, J., Adams, M. Z. A., & Khoveyni, M. (2019). Network DEA models for assessing urban water utility efficiency. *Utilities Policy*, *57*, 48–58.

Hollingsworth, B., & Smith, P. (2003). Use of ratios in data envelopment analysis. *Applied Economics Letters*, *10*(11), 733–735.

Kao, C., & Hwang, S.-N. (2008). Efficiency decomposition in two-stage data envelopment analysis: An application to non-life insurance companies in Taiwan. *European Journal of Operational Research*, *185*(1), 418–429.

Khoshnevis, P., & Teirlinck, P. (2018). Performance evaluation of R&D active firms. *Socio-Economic Planning Sciences*, *61*, 16–28.
Olesen, O. B., & Petersen, N. C. (2009). Target and technical efficiency in DEA: controlling for environmental characteristics. *Journal of Productivity Analysis*, *32*(1), 27–40.
Olesen, O. B., Petersen, N. C., & Podinovski, V. V. (2015). Efficiency analysis with ratio measures. *European Journal of Operational Research*, *245*(2), 446–462.
Olesen, O. B., Petersen, N. C., & Podinovski, V. V. (2017). Efficiency measures and computational approaches for data envelopment analysis models with ratio inputs and outputs. *European Journal of Operational Research*, *261*, 640–655.
Silva, M. C. A. (2018). Output-specific inputs in DEA: An application to courts of justice in Portugal. *Omega*, *79*, 43–53.
Thanassoulis, E., Boussofiane, A., & Dyson, R. . (1995). Exploring output quality targets in the provision of perinatal care in England using data envelopment analysis. *European Journal of Operational Research*, *80*(3), 588–607.

- We criticize some developed DEA models to deal with ratio data.
- We make modifications to explicitly overcome the flaws.
- We provide a case study in the education sector to validate our proposed approach.