FDGM(1,1) Model Based on Unified Fractional Grey Generation Operator

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Abstract: The grey prediction model DGM (1,1) uses the first-order accumulating generation sequence as the modeling sequence, and restores the result to the predicted value through the first-order reducing generation operator. The existing grey prediction models with fractional generation operators fail to unify the expression of the fractional accumulating generation operator and the reducing generation operator. By systematically studying the properties of the fractional accumulating operator and the reducing operator, and analyzing the sensitivity of the order value, an unified expression of the fractional operators is given. The FDGM (1,1) model with the unified fractional grey generation operator is established. The relationship between the order value and the modeling error distribution is studied. The DGM (1,1) model is a special case of the FDGM (1,1) model. The FDGM (1,1) model has a higher modeling accuracy and modeling adaptability than the DGM (1,1) by optimizing the order.

Keywords: grey prediction model; fractional operator; DGM(1,1)

1 Introduction

The grey system theory takes an uncertain system as the research object. This type of system has the characteristics of known partial information and unknown partial information, small sets of samples, and inadequate information. Through the generation and development of incomplete information, it extracts the valuable information and describes the operating behavior and evolution rules of the system, and to achieve quantitative prediction of its future changes [1-2]. As an essential part of the grey system theory, the grey prediction models have attracted significant attention in the research of modeling methods. Researchers have proposed discrete modeling methods [3-5], optimization methods of buffer operators [6-8], modeling methods of interval grey numbers [9-11], and modeling methods of delay functions [12-13]. In recent years, fractional grey operators and fractional grey prediction models have become research hotspots [14-28]. From the point of view of definition and connotation analysis, the grey accumulating generation operator in grey system theory and the sum operator in difference equation theory, the grey reducing generator, and the difference operator have all shared the same theoretical foundation.

As early as 1695, Leibniz wrote to L'Hospital to raise the following question: Can the meaning of derivatives with integer-order be generalized to derivatives with non-integer orders? L'Hospital wrote back and asked: What if the order will be 1/2? The question raised by Leibniz for a fractional derivative was an ongoing topic for more than 300 years. Until 1819 after 124
years, Lacroix firstly gave the correct answer: \( \frac{d^{1/2} x}{dx^{1/2}} = \frac{2}{\sqrt{\pi}} x^{1/2} \). Fractional calculus, as an extension of integer calculus in the set of real numbers, is more suitable for describing complex physical phenomena with genetic memory and path-dependent characteristics. Fractional calculus is widely used in the fields of anomalous diffusion, viscoelastic mechanics, rheology, chaos, cybernetics, signal processing, etc. [29]. Fractional differential equations are difficult to find analytical solutions. As the discretized form of fractional differential equations and the primary solution tool, fractional difference equations, and the numerical solution calculation methods have attracted extensive attention from researchers [30].

1.1 Fractional Difference Operator and Sum Operator

J. B. Diaz and T. J. Osler defined the \( \alpha \) th order forward difference operator.

**Definition 1.** For \( \alpha \in C \), the \( \alpha \) th order forward difference operator is

\[
\Delta^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x + \alpha - k)
\]

where,

\[
\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)k!}.
\]

This fractional forward difference operator does not satisfy the exponential law, and Granger and Joyeux defined the \( \alpha \) th order backward difference operator [32] to improve it,

\[
\nabla^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x-k) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)k} B^k f(x).
\]

The fractional difference operators defined by Diaz and Granger are in the form of infinite series, and there are rounding errors in numerical calculations. Henry L. Gray and Nien F. Zhang discretized the data and proposed the definitions of fractional sum operator and fractional difference operator.

**Definition 2.** For \( \alpha \in C \), let \( f(t) \), \( f(a) \), and \( a \in R \) with \( a > 0 \), the fractional sum operator is

\[
S^\alpha_a f(t) = \frac{\nabla^\alpha}{\Gamma(n+\alpha)} \sum_{k=0}^{\infty} (t-k+1)^n \Gamma(n+\alpha) \Gamma(n+\alpha) f(k),
\]

and the fractional difference operator is

\[
\nabla^\alpha_a f(t) = S^\alpha_a f(t) = \frac{\nabla^\alpha}{\Gamma(n-\alpha)} \sum_{k=0}^{\infty} (k+1)^n \Gamma(n) \Gamma(n) f(t-k).
\]

Based on the definitions of Henry L. Gray and other researchers, let \( \alpha = m = 1 \), \( a \in R \), and \( 0 < a < 1 \), Deekshitulu and Mohan gave the binomial expression definitions of fractional sum operator and fractional difference operator.

**Definition 3.** For \( \alpha \in R \), and \( a \geq 0 \), the definition of fractional sum operator [34] is
\[ \nabla^{-a} u(n) = \sum_{j=0}^{n} \binom{j+a-1}{j} u(n-j). \]  
\[ \nabla^{a} u(n) = \sum_{j=0}^{n} \binom{n-j+a-1}{n-j} u(j) - \binom{n-a-1}{n-1} u(0). \]

**Definition 4.** For \( \alpha \in \mathbb{R} \), and \( 0 < a \leq 1 \), the definition of fractional difference operator [34] is

Jinfa Cheng first defined the fractional sum operator as the following,

\[ \nabla^{-a} x(n) = \left[ \binom{v}{n} \right] x(n) = \sum_{r=0}^{n} \binom{v}{n-r} x(r), \quad v \in \mathbb{R}^+. \]

Then, he defined the fractional difference operator by the fractional sum operator,

\[ \nabla^{a} x(n) = \nabla^{a} \nabla^{-(m-\mu)} x(n) \nabla^{B} x(n), \quad m \in \mathbb{R}^+, \quad \mu \in \mathbb{R}^+. \]

According to the definitions given by Jinfa Cheng, to solve the fractional difference, we first need to address the \( \mu \) th (\( \mu \in \mathbb{R}^+ \)) sum, and then the \( m \) th (\( m \in \mathbb{Z}^+ \)) difference [35]. When Jinfa Cheng defined the fractional difference operator, he did not prove that the fractional difference operator and the fractional sum operator satisfy the exponential law. The analytical expressions of the fractional difference operator and sum operator were also not given. For \( \mu \in \mathbb{R} \) and \( \nu \in \mathbb{R} \), he inferred that the given fractional difference operator generally does not satisfy the exponential law, \( \nabla^{\mu} \nabla^{\nu} x(n) \neq \nabla^{\mu+\nu} x(n) \).

P. Bialiarsingh proposed the definition of the fractional difference operator as the following [36],

\[ (\Delta^{x})_{k} = \sum_{i=0}^{n} (-1)^{i} \frac{\Gamma(\alpha+1)}{i! \Gamma(\alpha-i+1)} x_{i+k}, \quad k \in \mathbb{N}. \]

### 1.2 Fractional Grey Accumulating Operator and Reducing Operator

Yu-Ran Liu defined the fractional accumulating operator using fractional Taylor expansion [24],

\[ f(x) = \frac{1}{\Gamma(ma+1)} \sum_{i=1}^{n} f(b-i+1) \cdot a_{i}. \]

Using binomial coefficient, Lifeng Wu defined the fractional accumulating generator as the following [25-26],

\[ x^{(r)}(k) = \sum_{i=1}^{n} \binom{k-i+r-1}{k-i} x^{(0)}(i), \quad 0 < r < 1, \quad k = 1, 2, \ldots, n. \]

Wu described \( \alpha^{(r)} x^{(0)}(k) = \alpha^{(r-1)} x^{(0)}(k) - \alpha^{(r-1)} x^{(0)}(k-1) \) as the \( r \) th (\( 0 < r < 1 \)) reducing operator. Firstly, the \( (1-r) \) th order accumulating sequence of the sequence \( X^{(0)} \) is calculated, and then the first order reducing sequence of the sequence \( X^{(1-r)} \) is derived. However, Wu did not prove whether the fractional grey accumulating operator and the reducing generation operator satisfy the exponential law, and only defined the fractional operators of order value within the range of \([0,1]\).
Xinping Xiao applied the coefficient matrix to define the fractional grey accumulating operator as the following,

\[ x^{(r)} = A^r x^{(0)} = (a^r_{ij})_{n \times n}, \quad (13) \]

where,

\[
(a^r_{ij})_{n \times n} = \begin{cases} 
\binom{r-1}{r-j-1} = \frac{(i-j+r-1)!}{(r-1)!(i-j)!} & \text{if } i \geq j \\
0 & \text{if } i < j 
\end{cases}.
\]  

Then through the inverse matrix \( A^r A^{-r} = E \), we can solve the coefficient matrix \( A^{-r} \) of the fractional reducing generation sequence [27-28].

In summary, the fractional difference operator and sum operator in the theory of fractional differential equations, the fractional accumulating generator and reducing generation operator in the grey system theory, have various definitions. These studies failed to give a unified analytical expression, nor did they prove whether the fractional operators satisfy the properties of the exchange law, the exponential law, and the reciprocity law. Wei Meng used the Gamma function \( \Gamma(n) \) to expand the integer accumulating operator and reducing generation operator to fractional operators. The 0th order, first order, and integer-order are all particular cases of fractional grey generating operators. It is proved that the fractional grey accumulating generation operator and the reducing generation operator satisfy the exchange law and the exponential law, and the accumulating operator and the reducing operator with the same order satisfy the reciprocity law [14-15]. Wei Meng’s research provides a theoretical basis for the fractional systems. However, Wei Meng failed to unify the expressions of the fractional grey accumulating generator and the reducing generator.

### 2 Fractional Grey Accumulating Generation Operator and Reducing Generation Operator

**Definition 5.** Assume that \( X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)) \) is the sequence of the raw data, \( X^{(i)} = (x^{(i)}(1), x^{(i)}(2), \ldots, x^{(i)}(n)) \) is the first order grey accumulating generation sequence of \( X^{(0)} \), where

\[ x^{(i)}(k) = \sum_{i=1}^{k} x^{(i)}(i), \quad k = 1, 2, \ldots, n. \]

We name \( x^{(i)}(k) = \sum_{i=1}^{k} x^{(i)}(i) \) as the first order grey accumulating generation operator [2], denoted as 1-AGO.

**Theorem 1.** Assume that \( X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)) \) is the sequence of the raw data, for \( r \in \mathbb{R}^+ \), \( X^{(r)} = (x^{(r)}(1), x^{(r)}(2), \ldots, x^{(r)}(n)) \) is the \( r \)th order grey accumulating generation sequence of \( X^{(0)} \), where
\[ x^{(r)}(k) = \sum_{i=1}^{k} \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)} x^{(0)}(i), \quad k = 1,2,\ldots,n. \quad (16) \]

We name \( x^{(r)}(k) = \sum_{i=1}^{k} \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)} x^{(0)}(i) \) as the fractional grey accumulating generation operator [14], denoted as \( r \)-AGO.

**Definition 6.** Assume that \( X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)) \) is the sequence of the raw data, \( X^{(-1)} = (x^{(-1)}(1), x^{(-1)}(2), \ldots, x^{(-1)}(n)) \) is the first order reducing generation sequence of \( X^{(0)} \), where \[ x^{(-1)}(k) = x^{(0)}(k) - x^{(0)}(k-1), \quad k = 1,2,\ldots,n. \quad (17) \]

We name \( x^{(-1)}(k) = x^{(0)}(k) - x^{(0)}(k-1) \) as the first order grey reducing generation operator[2], denoted as 1-RGO.

**Theorem 2.** Assume that \( X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)) \) is the sequence of the raw data, for \( r \in \mathbb{R}^+ \), \( X^{(-r)} = (x^{(-r)}(1), x^{(-r)}(2), \ldots, x^{(-r)}(n)) \) is the \( r \)th order grey reducing generation sequence of \( X^{(0)} \), where
\[ x^{(-r)}(k) = \sum_{i=1}^{k} (-1)^i \frac{\Gamma(r+1)}{\Gamma(i+1)\Gamma(r-i+1)} x^{(0)}(k-i), \quad k = 1,2,\ldots,n. \quad (18) \]

We name \( x^{(-r)}(k) = \sum_{i=1}^{k} (-1)^i \frac{\Gamma(r+1)}{\Gamma(i+1)\Gamma(r-i+1)} x^{(0)}(k-i) \) as the fractional grey reducing generation sequence[14], denoted as \( r \)-RGO.

**Theorem 3.** Assume that \( X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)) \) is the sequence of the raw data, for \( p \in \mathbb{R}^+ \), and \( q \in \mathbb{R}^+ \), \( X^{(p)} \) is the \( p \)th order grey accumulating generation sequence of \( X^{(0)} \), \( X^{(-q)} \) is the \( q \)th order grey reducing generation sequence of \( X^{(0)} \), \( (X^{(p)})^{(-q)} \) is the \( q \)th order grey reducing generation sequence of \( X^{(p)} \), \( (X^{(-q)})^{(p)} \) is the \( p \)th order grey accumulating generation sequence of \( X^{(-q)} \). The fractional grey accumulating operator and reducing operator satisfy the exchange law, and the exponential law [14].
\[ X^{(p-q)} = (X^{(p)})^{(-q)} = (X^{(-q)})^{(p)} \quad (19) \]

1) If \( p-q > 0 \), \( X^{(p-q)} \) is the \( (p-q) \)th order grey accumulating generation sequence of \( X^{(0)} \);
2) if \( p-q < 0 \), \( X^{(p-q)} \) is the \( (q-p) \)th order grey reducing generation sequence of \( X^{(0)} \);
3) if \( p-q = 0 \), the same order of grey accumulating generation operator and reducing operator satisfy the reciprocity law.

### 3 Uniform of Fractional Grey Generation Operators

The definitions of fractional grey accumulating generator and fractional grey reducing generator are based on the Gamma function \( \Gamma(n) \). \( \Gamma(n) \) is an extension of the integer factorial in the real number set, \( \Gamma(n+1) = \int_0^\infty e^{-t}t^n dt = n! \). The domain of the Gamma function is \( n \in \mathbb{R} \).
and \( n \not\in \{-1,-2,-3,\ldots\} \).

Literature [22] has discussed that due to the domain of the Gamma function, the negative integer factorial is not defined, and the exchange law and exponential law of the fractional grey reducing generator cannot be theoretically proven. By numerical simulation, Using Matlab, any sequence and any order can verify Theorem 3 is correct.

### 3.1 Verification by Numerical Method

Example 1. According to the data of the "Statistical Bulletin of the National Economic and Social Development of the People's Republic of China 2019", the total retail sales of consumer goods in China from 2015 to 2019 (unit: trillion yuan) is

\[
X = (30.093,33.232,36.626,38.099,41.165)
\]

The grey accumulating generation sequences of the sequence \( X \) with different orders are shown in Table 1. The grey reducing generation sequences of the sequence \( X \) with varying values of the order are shown in Table 2. Comparing the data in Table 1 and Table 2, we can conclude that the mean absolute percentage error (MAPE) is 0.000000% between Table 1 and Table 2. Due to the domain of the Gamma function, all the negative integer order grey accumulating generation sequences and grey reducing generation sequences do not exist. Considering the calculation error, the expressions of fractional accumulating generation operator and reducing generation operator are equivalent to each other. Any sequence and any order can verify that this conclusion is correct.

| Table 1. Grey Accumulating Generation Sequences of \( X \) |
|------------------------|----------------|----------------|----------------|----------------|
| \( r \) | \( k = 1 \) | \( k = 2 \) | \( k = 3 \) | \( k = 4 \) | \( k = 5 \) |
|---|---|---|---|---|
| \( r = 0 \) | 30.093 | 33.232 | 36.626 | 38.099 | 41.165 |
| \( r = 1/4 \) | 30.093 | 40.755 | 49.636 | 55.975 | 63.172 |
| \( r = 1/3 \) | 30.093 | 43.263 | 54.391 | 62.894 | 72.082 |
| \( r = 1/2 \) | 30.093 | 48.279 | 64.527 | 78.278 | 92.563 |
| \( r = 2/3 \) | 30.093 | 53.294 | 75.499 | 95.839 | 116.950 |
| \( r = 3/4 \) | 30.093 | 55.802 | 81.299 | 105.480 | 130.740 |
| \( r = 1 \) | 30.093 | 63.325 | 99.951 | 138.050 | 179.210 |
| \( r = 1.25 \) | 30.093 | 70.848 | 120.480 | 176.460 | 239.630 |
| \( r = 1.5 \) | 30.093 | 78.371 | 142.900 | 221.180 | 313.740 |
| \( r = -1/4 \) | 30.093 | 25.709 | 25.497 | 24.181 | 25.258 |
| \( r = -1/3 \) | 30.093 | 23.201 | 22.205 | 20.340 | 21.106 |
| \( r = -2/3 \) | 30.093 | 13.170 | 11.128 | 8.503 | 9.188 |
| \( r = -3/4 \) | 30.093 | 10.662 | 8.881 | 6.339 | 7.198 |
| \( r = -1 \) | NaN | NaN | NaN | NaN | NaN |
| \( r = -1.25 \) | 30.093 | -4.384 | -0.212 | -1.316 | 1.077 |
| \( r = -1.5 \) | 30.093 | -11.907 | -1.937 | -2.497 | 0.534 |
3.2 Unify the Expression of Fractional Grey Generation Operators

According to the result of numerical verification in Example 1, the fractional grey accumulating generator can be used as the unified expression of the fractional grey generation operators. For \( r \in \mathbb{R} \), if \( r > 0 \), the expression is the grey accumulating generator. If \( r < 0 \), the expression is the grey reducing generation operator. If \( r = 0 \), the expression is the sequence of the raw data. The grey accumulating generator with negative integer-order is not defined, and it can be calculated by the grey reducing generator with positive integer order.

**Definition 7.** Assume that \( X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)) \) is the sequence of the raw data, for \( r \in \mathbb{R} \), \( X^{(r)} = (x^{(r)}(1), x^{(r)}(2), \ldots, x^{(r)}(n)) \) is the \( r \) th order grey generation operator of \( X^{(0)} \), where

\[
x^{(r)}(k) = \begin{cases} \sum_{i=1}^{k} \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)} x^{(0)}(i) & \text{for } r \in \mathbb{R} \setminus \mathbb{Z}^- \cap \mathbb{R} \setminus \mathbb{N}, \quad k = 1, 2, \ldots, \; n, \\ \sum_{i=0}^{k-1} (-1)^i \frac{\Gamma(r+i)}{\Gamma(i+1)\Gamma(r-i+1)} x^{(0)}(k-i) & \text{for } r \in \mathbb{Z}^- \setminus \mathbb{N} \setminus \{0\}, \quad k = 1, 2, \ldots, \; n. \end{cases}
\]

We name the Formula (20) as the fractional grey generation sequence, denoted as \( r \)-FGO.

1) If \( r > 0 \), \( X^{(r)} \) is the \( r \) th order grey accumulating generator sequence of \( X^{(0)} \);
2) if \( r < 0 \), \( X^{(r)} \) is the \( r \) th order grey reducing generation sequence of \( X^{(0)} \);
3) if \( r = 0 \), \( X^{(r)} \) is the \( X^{(0)} \).

3.3 Sensitivity Analysis of Negative Integer grey Generator

<table>
<thead>
<tr>
<th>( r )</th>
<th>( k = 1 )</th>
<th>( k = 2 )</th>
<th>( k = 3 )</th>
<th>( k = 4 )</th>
<th>( k = 5 )</th>
<th>MAPE with Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r = 0 )</td>
<td>30.093</td>
<td>33.232</td>
<td>36.626</td>
<td>38.099</td>
<td>41.165</td>
<td>30.093</td>
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<tr>
<td>( r = 1/4 )</td>
<td>30.093</td>
<td>25.709</td>
<td>25.497</td>
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</tr>
<tr>
<td>( r = 1/3 )</td>
<td>30.093</td>
<td>23.201</td>
<td>22.205</td>
<td>20.340</td>
<td>21.106</td>
<td>0.000000%</td>
</tr>
<tr>
<td>( r = 1/2 )</td>
<td>30.093</td>
<td>18.185</td>
<td>16.248</td>
<td>13.751</td>
<td>14.285</td>
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</tr>
<tr>
<td>( r = 2/3 )</td>
<td>30.093</td>
<td>13.170</td>
<td>11.128</td>
<td>8.503</td>
<td>9.188</td>
<td>0.000000%</td>
</tr>
<tr>
<td>( r = 3/4 )</td>
<td>30.093</td>
<td>10.662</td>
<td>8.881</td>
<td>6.339</td>
<td>7.198</td>
<td>0.000000%</td>
</tr>
<tr>
<td>( r = 1 )</td>
<td>30.093</td>
<td>3.139</td>
<td>3.394</td>
<td>1.473</td>
<td>3.066</td>
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</tr>
<tr>
<td>( r = 1.25 )</td>
<td>30.093</td>
<td>-4.384</td>
<td>-0.212</td>
<td>-1.316</td>
<td>1.077</td>
<td>0.000000%</td>
</tr>
<tr>
<td>( r = 1.5 )</td>
<td>30.093</td>
<td>-11.907</td>
<td>-1.937</td>
<td>-2.497</td>
<td>0.534</td>
<td>0.000000%</td>
</tr>
<tr>
<td>( r = -1/4 )</td>
<td>30.093</td>
<td>40.755</td>
<td>49.636</td>
<td>55.975</td>
<td>63.172</td>
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<tr>
<td>( r = -1/3 )</td>
<td>30.093</td>
<td>43.263</td>
<td>54.391</td>
<td>62.894</td>
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<tr>
<td>( r = -1/2 )</td>
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<td>48.279</td>
<td>64.527</td>
<td>78.278</td>
<td>92.563</td>
<td>0.000000%</td>
</tr>
<tr>
<td>( r = -2/3 )</td>
<td>30.093</td>
<td>53.294</td>
<td>75.499</td>
<td>95.839</td>
<td>116.950</td>
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</tr>
<tr>
<td>( r = -3/4 )</td>
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<td>55.802</td>
<td>81.299</td>
<td>105.480</td>
<td>130.740</td>
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</tr>
<tr>
<td>( r = -1 )</td>
<td>NaN</td>
<td>NaN</td>
<td>NaN</td>
<td>NaN</td>
<td>NaN</td>
<td>NaN</td>
</tr>
<tr>
<td>( r = -1.25 )</td>
<td>30.093</td>
<td>70.848</td>
<td>120.480</td>
<td>176.460</td>
<td>239.630</td>
<td>0.000000%</td>
</tr>
<tr>
<td>( r = -1.5 )</td>
<td>30.093</td>
<td>78.371</td>
<td>142.900</td>
<td>221.180</td>
<td>313.740</td>
<td>0.000000%</td>
</tr>
</tbody>
</table>
Using the data sequence of Example 1, the fractional grey generation operator of Formula (18) and the fractional grey generation operator of Formula (20) are applied respectively to calculate 1, 0.9, 1.1, 0.99, 1.01, 0.999, 1.001, 0.9999, 1.0001 order accumulating generation sequences. The calculation results are shown in Table 3. The analysis can be concluded as follows.

1) The first-order reducing generation sequence should be calculated using Formula (18) rather than Formula (20).

2) Considering the error, even if the accuracy of the order value is 0.0001, the result of the grey generation sequence is the same as the fractional grey reducing generation operator. That is, except for the integer reducing generation sequence, Formula (20) can completely replace Formula (18).

3) The accuracy of the order value affects the accuracy of the grey reducing generation sequences. When the precision of order value is 0.1, the deviation of the reducing generation sequences value is as high as 62.283%; when the precision of order is 0.01, the difference of the reducing generation sequences is reduced to 5.8%; when the accuracy of order is 0.0001, the deviation of the reducing generation sequence is less than 0.06%.

4) Just as negative integers are "coincidences" and exceptional cases of negative real number sets, integer grey reducing generation sequences are also "coincidences" and extraordinary examples of positive real order grey accumulation generation sequences. Even if the precision of order value is under 0.0001, the Formula (20) can also replace Formula (18). Therefore, in practical applications, the unified expression of the fractional grey generator can be simplified to formula (21).

\[
x^{(r)}(k) = \sum_{i=0}^{k} \frac{\Gamma(r + k - i)}{\Gamma(r) \Gamma(k - i + 1)} x^{(0)}(i), \quad k = 1, 2, \ldots, n.
\]  

Table 3. Sensitivity Analysis of Integer grey Accumulating Generation Sequence

<table>
<thead>
<tr>
<th>Reducing Generation Sequences</th>
<th>Formula</th>
<th>MAPE with First Reducing Sequences</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Formula(18)</td>
<td>k = 1</td>
</tr>
<tr>
<td></td>
<td>Formula(20)</td>
<td>NaN</td>
</tr>
<tr>
<td>0.9</td>
<td>Formula(18)</td>
<td>30.093</td>
</tr>
<tr>
<td></td>
<td>Formula(20)</td>
<td>NaN</td>
</tr>
<tr>
<td>1.1</td>
<td>Formula(18)</td>
<td>30.093</td>
</tr>
<tr>
<td></td>
<td>Formula(20)</td>
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<td>0.99</td>
<td>Formula(18)</td>
<td>30.093</td>
</tr>
<tr>
<td></td>
<td>Formula(20)</td>
<td>30.093</td>
</tr>
<tr>
<td>1.01</td>
<td>Formula(18)</td>
<td>30.093</td>
</tr>
<tr>
<td></td>
<td>Formula(20)</td>
<td>30.093</td>
</tr>
<tr>
<td>0.999</td>
<td>Formula(18)</td>
<td>30.093</td>
</tr>
<tr>
<td></td>
<td>Formula(20)</td>
<td>30.093</td>
</tr>
</tbody>
</table>
### 3.4 Properties of Unified Fractional Grey Generation Operator

**Proposition 1.** For $|r| \in (0,1)$, the fractional grey generation operator satisfies the principle of newer information priority.

**Prove**

According to the expression of $x^{(r)}(k)$,

the coefficient of $x^{(0)}(i)$ is $a_i = \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)}$, $k = 1,2,\ldots,n$,

and the coefficient of $x^{(0)}(i-1)$ is $a_{i-1} = \frac{\Gamma(r+k-i+1)}{\Gamma(k-i+2)\Gamma(r)}$, $k = 1,2,\ldots,n$.

\[
\frac{a_i}{a_{i-1}} = \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)} \cdot \frac{\Gamma(k-i+1)\Gamma(r)}{\Gamma(r+k-i+1)} = \frac{r+k-i}{k-i+1} = 1 + \frac{1-r}{r-i+k}.
\]

If $|r| \in (0,1)$, $i \leq k$, $\frac{a_i}{a_{i-1}} = 1 + \frac{1-r}{r-i+k} > 1$.

In the expression of $x^{(r)}(k)$, $x^{(0)}(i)$ has a greater weight than $x^{(0)}(i-1)$. So it satisfies the principle of newer information priority.

**Proposition 2.** For $|r| > 1$, the fractional grey generation operator satisfies the principle of older information priority.

**Prove**

\[
\frac{a_i}{a_{i-1}} = \frac{\Gamma(k-i+2)\Gamma(r)}{\Gamma(r+k-i+1)} \cdot \frac{\Gamma(r+k-i+1)}{\Gamma(k-i+1)\Gamma(r)} = R + \frac{r-1}{r+k-i}.
\]

If $|r| > 1$, and $i \leq k$, $\frac{a_i}{a_{i-1}} = 1 + \frac{r-1}{r+k-i} > 1$.

In the expression of $x^{(r)}(k)$, $x^{(0)}(i-1)$ has a greater weight than $x^{(0)}(i)$. So it satisfies the principle of older information priority.

**Proposition 3.** For $|r| = 1$, the weights of the older and newer information in the first order
grey generation operator are equal.

Prove

If \(|r| = 1, \ a_{i-1} = a_i\).

In the expression of \(x^{(r)}(k), x^{(0)}(i-1)\) and \(x^{(0)}(i)\) have the same weight coefficient of 1.

That is, newer and older information is equally important.

3.5 Examples

Example 2. Assume that \(X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(k), \ldots, x^{(0)}(n))\) is the sequence of the raw data,

\[
X^{(r)} = \begin{bmatrix}
    x^{(r)}(1) \\
    x^{(r)}(2) \\
    \vdots \\
    x^{(r)}(k) \\
    \vdots \\
    x^{(r)}(n)
\end{bmatrix} = D_n^{(r)} \cdot X = D_n^{(r)} \cdot \begin{bmatrix}
    x^{(0)}(1) \\
    x^{(0)}(2) \\
    \vdots \\
    x^{(0)}(k) \\
    \vdots \\
    x^{(0)}(n)
\end{bmatrix}.
\]

If \(n = 5\), we can get the following results,

\[
D_5^{(1/2)} = \begin{bmatrix}
    1 & 0 & 0 & 0 & 0 \\
    \frac{1}{2} & 1 & 0 & 0 & 0 \\
    \frac{3}{8} & \frac{1}{2} & 1 & 0 & 0 \\
    \frac{5}{16} & \frac{3}{8} & \frac{1}{2} & 1 & 0 \\
    \frac{1}{128} & \frac{5}{16} & \frac{3}{8} & \frac{1}{2} & 1 \\
\end{bmatrix},
\]

\[
D_5^{(1/2)} = \begin{bmatrix}
    1 & 0 & 0 & 0 & 0 \\
    -\frac{1}{2} & 1 & 0 & 0 & 0 \\
    -\frac{1}{8} & -\frac{1}{2} & 1 & 0 & 0 \\
    -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{2} & 1 & 0 \\
    -\frac{1}{128} & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{2} & 1 \\
\end{bmatrix},
\]

\[
D_5^{(1)} = \begin{bmatrix}
    1 & 0 & 0 & 0 & 0 \\
    1 & 1 & 0 & 0 & 0 \\
    1 & 1 & 1 & 0 & 0 \\
    1 & 1 & 1 & 1 & 0 \\
    1 & 1 & 1 & 1 & 1 \\
\end{bmatrix},
\]

\[
D_5^{(1)} = \begin{bmatrix}
    1 & 0 & 0 & 0 & 0 \\
    -1 & 1 & 0 & 0 & 0 \\
    0 & -1 & 1 & 0 & 0 \\
    0 & 0 & -1 & 1 & 0 \\
    0 & 0 & 0 & -1 & 1 \\
\end{bmatrix}.
\]

4 Modeling Method of FDGM(1,1)

Definition 8 Assume that \(X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n))\) is the sequence of the raw data,

\[
Z^{(r)} = (z^{(r)}(2), z^{(r)}(3), \ldots, z^{(r)}(n))\]

is the adjacent mean generation sequence of \(X^{(r)}\), where

\[
z^{(r)}(k) = \frac{x^{(r)}(k) + x^{(r)}(k-1)}{2}, \quad k = 2, 3, \ldots, n.
\]

We name

\[
x^{(r-1)}(k) + az^{(r)}(k) = b
\]

as a fractional discrete grey model, denoted as FDGM(1,1).

1) If \(r = 0\), it is the direct modeling DGM(1,1);
2) If \( r = 1 \), it is the DGM(1,1).

**Theorem 4** Assume that \( X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n)) \) is the sequence of the raw data, for \( r \in \mathbb{R}^* \), \( X^{(r)} \) is the fractional grey accumulating generation sequence of \( X^{(0)} \), \( X^{(r)} \) is the fractional grey reducing generation sequence of \( X^{(0)} \), \( Z^{(r)} \) is the adjacent mean generation sequence of \( X^{(r)} \), the parameter vector \( \hat{a} = [a, b]^T \) of the FDGM(1,1) \( x^{(r-1)}(k) + az^{(r)}(k) = b \) can be calculated by the least square method, 

\[
\hat{a} = (B^T B)^{-1} B^T Y
\]

where

\[
Y = \begin{bmatrix}
x^{(r-1)}(2) \\
x^{(r-1)}(3) \\
\vdots \\
x^{(r-1)}(n)
\end{bmatrix}, \quad B = \begin{bmatrix}
-z^{(r)}(2) & 1 \\
-z^{(r)}(3) & 1 \\
\vdots & \vdots \\
-z^{(r)}(n) & 1
\end{bmatrix}
\]

Furthermore,

\[
Y = \left[ \begin{array}{c}
(r-1)x^{(0)}(1) + x^{(0)}(2) \\
\frac{r(r-1)}{2}x^{(0)}(1) + (r-1)x^{(0)}(2) + x^{(0)}(3) + \sum_{i=1}^{r-3} \frac{\Gamma(r+n-i)}{\Gamma(n-i+1)\Gamma(r)} x^{(0)}(i) \\
\vdots \\
\frac{1}{2} \sum_{i=1}^{r-3} \frac{\Gamma(r+n-i)}{\Gamma(n-i+1)\Gamma(r)} x^{(0)}(i) \\
\end{array} \right]
\]

\[
B = \left[ \begin{array}{c}
\frac{-1}{2} \left[ (r+1)x^{(0)}(1) + x^{(0)}(2) \right] \\
\frac{-1}{2} \left[ \frac{r(r+3)}{2}x^{(0)}(1) + (r+1)x^{(0)}(2) + x^{(0)}(3) \right] \\
\vdots \\
\frac{-1}{2} \sum_{i=1}^{r-3} \frac{\Gamma(r+n-i)}{\Gamma(n-i+1)\Gamma(r)} x^{(0)}(i) \\
\end{array} \right] + 1
\]

**Theorem 5** Assume that \( B, Y, \) and \( \hat{a} \) are defined as Theorem 4, \( \hat{a} = [a, b]^T = (B^T B)^{-1} B^T Y \).

1) The time response sequence of FDGM(1,1) \( x^{(r-1)}(k) + az^{(r)}(k) = b \) is,

\[
\hat{x}^{(r)}(k) = (x^{(0)}(1) - \frac{b}{a})e^{-a(k-1)} + \frac{b}{a}, \quad k = 2, 3, \ldots, n
\]

2) The restore values are

\[
\hat{X}^{(0)}(k) = (\hat{x}^{(r)})^{(r-1)}(k) = \begin{cases} 
   x^{(0)}(1) & k = 1 \\
   \sum_{i=0}^{k} \frac{\Gamma(-r+k-i)}{\Gamma(k-i+1)\Gamma(-r)} x^{(0)}(i) \hat{x}^{(r)}(k-i) & k = 2, 3, \ldots, n
\end{cases}
\]

5 Modeling Accuracy and Order Optimization

5.1 Relationship Between Modeling Accuracy and Order Value

Taking the sequence of total retail sales of consumer goods in China from 2015 to 2019 (unit: trillion yuan) as the example, the relationship between the modeling accuracy and the order value
of the FDGM(1,1) is analyzed.

\[ X = (30.093, 33.232, 36.626, 38.099, 41.165) \]

The order value and mean absolute percentage error (MAPE) of FDGM(1,1) for \( X \) are shown in Figure 1. If \( r = 0 \), it is the direct modeling DGM(1,1), with MAPE=0.843%. If \( r = 1 \), it is the DGM (1,1), with MAPE=0.985%. As can be seen from Figure 1, the MAPE of the DGM(1,1) with order \( r = 1 \) is not the smallest. For the data in this example, MAPE has two minimum values, MAPE=0.813% for \( r = 0.05 \), and MAPE=0.828% for \( r = 0.84 \).

As the order value increases in the interval \([0, 1.5]\), the value of MAPE firstly decreases to 0.813%, then increases to 1.157 %, and then falls to 0.828%, and finally continues to increase rapidly when \( a > 1 \). The relationship between modeling accuracy and order value is shown in Figure 1. From Figure 1, we can see that the DGM(1,1) is a particular case of the FDGM(1,1) model, and the optimized order is generally not "coincidence" as 1.

![Figure 1. Relationship Between Modeling Accuracy and Order Value](image)

### 5.2 Optimization Algorithm of Order

The optimal value of the order for FDGM(1,1) considering the minimum MAPE can be calculated by solving the following optimization problem:

\[
\min f(r) = \frac{1}{n-1} \sum_{k=2}^{n} \frac{|x^{(0)}(k) - \hat{x}^{(0)}(k)|}{x^{(0)}(k)}, \quad r \in \mathbb{R}^{+}
\]  

(29)

Using a particle swarm optimization algorithm [14], for \( r = 0.05 \), the FDGM(1,1) has the minimum MAPE=0.813%, and the modeling accuracy is higher than the DGM(1,1) with MAPE=0.985%. For the order, \( r = 0, 0.05, 0.5, 1 \), the modeling accuracy of the FDGM(1,1) is
shown in Table 4.

### Table 4 Modeling Accuracy of FDGM(1,1) for Total Retail Sales of Consumer Goods in China

<table>
<thead>
<tr>
<th>Year</th>
<th>$x^{(0)}(k)$</th>
<th>$r = 0$</th>
<th>$r = 0.05$</th>
<th>$r = 0.5$</th>
<th>$r = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{x}^{(0)}(k)$</td>
<td>$\Delta_1 (%)$</td>
<td>$\hat{x}^{(0)}(k)$</td>
<td>$\Delta_1 (%)$</td>
<td>$\hat{x}^{(0)}(k)$</td>
</tr>
<tr>
<td>2015</td>
<td>30.093</td>
<td>30.093</td>
<td>33.333</td>
<td>0.303</td>
<td>33.237</td>
</tr>
<tr>
<td>2016</td>
<td>33.232</td>
<td>33.232</td>
<td>36.227</td>
<td>1.090</td>
<td>36.256</td>
</tr>
<tr>
<td>2017</td>
<td>36.626</td>
<td>36.626</td>
<td>38.812</td>
<td>1.871</td>
<td>38.868</td>
</tr>
<tr>
<td>2018</td>
<td>38.099</td>
<td>38.099</td>
<td>41.121</td>
<td>1.071</td>
<td>41.08</td>
</tr>
<tr>
<td>2019</td>
<td>41.165</td>
<td>41.165</td>
<td>41.121</td>
<td>1.017</td>
<td>41.08</td>
</tr>
<tr>
<td></td>
<td>MAPE</td>
<td>0.843%</td>
<td>0.813%</td>
<td>1.156%</td>
<td>0.985%</td>
</tr>
</tbody>
</table>

### 6 Conclusions

In this paper, based on the fractional accumulating generation operator and the fractional reducing generation operator, a unified expression of the fractional grey generation operator is studied, and the sensitivity of the fractional grey generation operator is discussed. Due to the domain of the Gamma function, except for the negative integer value of the undefined field, the fractional grey accumulating generator and the reducing generator are equivalent to each other. In actual prediction problems, integer values are "coincidences" and particular cases of real numbers. Numerical simulations show that order values have high sensitivity. Therefore, the fractional grey accumulating operator and the reducing operator can be simplified to a unified expression.

Based on the unified fractional grey generation operator, FDGM(1,1) is established. The modeling method of the FDGM(1,1) is studied, and the relationship between modeling accuracy and order value is discussed. The result shows that the DGM(1,1) is a particular case of the FDGM(1,1) with $r = 1$, directly modeling DGM(1,1) is a particular case of the FDGM(1,1) with $r = 0$. More generally, the order of FDGM(1,1) with the minimum MAPE will be in the range of $[0, 1]$. The unification of the fractional grey accumulating generator and the reducing generation operator is of considerable significance to the research of fractional systems. The FDGM(1,1) is used to improve the simulation accuracy of the grey prediction model and expand the application range of grey prediction models. The fitting accuracy and application range of FDGM(1,1) for different sequences need further study.

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References
[12] Li Cong, Xie Xiuping. GM(1,1|τ_i) prediction model with time-varying delay function and its application[J]. Systems Engineering Theory & Practice, 2019, 39(06):1535-1549.


