Robust topology optimization under material and loading uncertainties using an evolutionary structural extended finite element method

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Abstract

This research presents a novel algorithm for robust topology optimization of continuum structures under material and loading uncertainties by combining an evolutionary structural optimization (ESO) method with an extended finite element method (XFEM). Conventional topology optimization approaches (e.g. ESO) often require additional post-processing to generate a manufacturable topology with smooth boundaries. By adopting the XFEM for boundary representation in the finite element (FE) framework, the proposed method eliminates this time-consuming post-processing stage and produces a more accurate evaluation of the elements along the design boundary for ESO-based topology optimization methods. A truncated Gaussian random field (without negative values) using a memory-less translation process is utilized for the random uncertainty analysis of the material property and load angle distribution. The superiority of the proposed method over Monte Carlo, solid isotropic material with penalization (SIMP) and polynomial chaos expansion (PCE) using classical finite element method (FEM) is demonstrated via two practical examples with compliances in material uncertainty and loading uncertainty improved by approximately 11% and 10%, respectively. The novelty of the present method lies in the following two aspects: (1) this paper is among the first to use the XFEM in studying the robust topology optimization under uncertainty; (2) due to the adopted XFEM for boundary elements in the FE framework, there is no need for any post-processing techniques. The effectiveness of this method is justified by the clear and smooth boundaries obtained.

Keywords: Robust topology optimization, Continuous structure, Uncertainty, Evolutionary structural optimization, Extended Finite Element Method

1. Introduction

In real applications, uncertainty is inevitably observed due to insufficient knowledge, production errors, changeable environment, etc. Herein, uncertainty refers to epistemic uncertainty that occurs due to inadequate knowledge and data (another type of uncertainty due to the inherent randomness of data is known as aleatoric uncertainty, which is out of the scope of this research). Although structural topology is possible, the design may not be feasible without considering uncertainties as the optimal structure may become sensitive and vulnerable. In other words, deterministic design can lead to optimized structural topology, but the only thing is that the structure may fail when subject to properties or performance variations. Therefore, there is a strong need to consider the effect of uncertainty on the optimal topology in structural design.

To account for different uncertainties in topology optimization, two major paradigms, robust topology optimization (RTO) and reliability-based topology optimization (RBTO), are often used.
RTO aims to optimize the objective performance while simultaneously minimizing its sensitivity with respect to uncertain variables such as the load, material property and geometry. RBTO seeks a design that can achieve a targeted probability of failure and thus ensures that the conditions that may lead to catastrophe are unlikely [1,2]. More information on the applications and numerical techniques of the RBTO can be referred to [3-7]. This paper is concerned only with the RTO under material and loading uncertainties.

In the area of the RTO under uncertainties, the RTO problems can be generally categorized into four types in terms of the uncertain variables. The first category is mainly focused on loading uncertainty (LU) [8-11]. Zhao et al. [12] studied the RTO under LU based on the stochastic collocation method combined with the tensor product network and Smolyak sparse grid that transformed the robust formulation into a weighted multiple loading deterministic problem at the collocation points. Wang and Gao [13] investigated the topology optimization (TO) of a continuous structure (which is usually formed from a surface element and does not have a distinct joint) under LU in its application position with a non-probabilistic approach, while Peng et al. [14] used a perturbation method to calculate the mean and standard deviation of minimum compliance for similar RTO problems. The second category of current RTO problems is concerned with material uncertainty (MU). Tootkaboni et al. [15] investigated the topology optimization of continuum structures under uncertainty in material properties by combining the polynomial chaos expansion (PCE) with a deterministic topology optimization technique. Chen et al. [16] proposed a robust shape and topology optimization approach with random field uncertainty in loading and material properties for structures and compliant mechanism designs based on the level set method (LSM). The third category is related to geometric uncertainty (GU) [17,18]. Chen and Chen [19] developed a level set-based approach for shape and topology optimization, in which GU was modelled by combining level set equation with a random boundary velocity field characterized using the Karhunen-Loeve expansion (KLE), and multivariate Gauss quadrature was employed to facilitate shape sensitivity analysis by transforming the RTO problem into a weighted summation of deterministic TO problems. Keshavarzzadeh et al. [20] introduced a systematic approach for topology optimization under uncertainties in loading and geometry that integrated non-intrusive PCE with design sensitivity analysis for reliability-based and robust topology optimization. In addition, the combination of different uncertainties, called hybrid uncertainty (HU), has recently emerged in the RTO area with various applications [21-23]. Latifi Rostami and Ghoddosian [24-26] investigated the RTO under hybrid uncertainties for continuous structures using stochastic collocation methods. Wu et al. [27,28] proposed the RTO methods for the computational design of metamaterials and structures under hybrid uncertainties with both interval and random variables.

The research efforts expanded so far on the RTO problems under uncertainties are mainly based on classical finite element methods. However, optimal topologies obtained from these conventional TO methods usually have unsmooth boundaries that are unsuitable for design-dependent problems (e.g. additive manufacturing) and often require further interpretation using other techniques. In this regard, the XFEM combined with the LSM has been recently introduced to the structural TO problems [29-34]. The problem boundary is usually defined with LSM and structural response predicted by XFEM. Abdi et al. [35,36] investigated the structural TO problems by combining the XFEM with an evolutionary optimization algorithm, in which an isoline design approach was used for boundary representation, and XFEM was used for improving the accuracy of FE solutions on the boundary during the optimization process. Lang et al. [37,38] proposed an extended stochastic finite element method by combing the XFEM with a PCE for heat transfer modelling in composite and heterogeneous materials with uncertain inclusion geometry.
From the preceding analysis, it can be found that, although the XFEM has been implemented to the deterministic TO with a variety of applications, very few investigations on the RTO problems under uncertainty using XFEM have been reported. On the other hand, most of the deterministic TO problems presented above uses the LSM to define the design boundary. Although LSM is accurate and efficient, it may mathematically complicate the optimization process. In this research, the XFEM has been used as an alternative that does not have the limitations of the LSM while improving the poor accuracy of the classical FEM in representing the boundary. More information on other numerical methods for boundary representation and treatment can be referred to [39-42]. To this end, the primary motivation of this paper is to resolve a shortcoming in most ESO-based methods and develop a robust ESO topology optimization method with a better and accurate evaluation of the elements along the design boundary.

The remainder of this paper is organized as follows. The detailed methodology of the robust topology optimization using an evolutionary structural extended finite element method (named ES-XFEM) is elaborated in Section 2. The applicability of the proposed method in the topology optimization of robust minimum compliance is shown in Section 3 with numerical examples. Finally, conclusions are presented in Section 4.

2. Methodology

2.1. Deterministic topology optimization

The ESO method is a deterministic topology design method based on the concept of gradually removing inefficient materials from the structure. Since the final design is wholly stressed, the ESO method uses numerical solvers such as finite element analysis. The bi-directional evolutionary structural optimization (BESO) method is similar to the ESO method, except that it makes it possible to remove and add materials simultaneously. The prototype is the entire design domain in the ESO method, where the inefficient elements are removed in multiple iterations. In contrast, in the BESO method, only the aspects between the forces and supports are present, and the element is added to or subtracted from the structure in subsequent iterations [43,44].

In this research, the BESO method has been used to perform topology optimization. The design criterion in this method is strain energy. The problem of compliance optimization (minimization) using the BESO method is expressed as follows

$$\text{minimize: } C = \frac{1}{2} U^T K U = \frac{1}{2} \sum_{e=1}^{n} u_e^T k_e u_e$$

subject to:

$$\frac{\sum_{e=1}^{n} v_e \rho_e}{V^0} = V_c$$

$$K U = F$$

$$\rho_e = \rho_{\text{min}} \text{ or } 1$$

Where $C$ is the compliance, $K$ is the global stiffness matrix, $U$ is the global displacement vector, and $F$ the force vector. $n$, $u_e$ and $k_e$ represent the total number of elements, element displacement vector and element stiffness matrix, respectively. $v_e$, $\rho_e$, $V_c$ and $V^0$ are element volume, design variable, prescribed volume fraction and design domain volume, respectively. $\rho_{\text{min}}$ is a minimum (nonzero) relative density used for the void elements to avoid the singularity. The sensitivity of
the objective function (1) to the design variable is calculated from the following equation

$$S_e = \frac{\partial C}{\partial \rho_e} = \frac{\partial}{\partial \rho_e} \left( \frac{1}{2} \sum_{e=1}^{n} u_e^T k_e u_e \right)$$ (2)

The element stiffness matrix $k_e$ is defined in the same way as that in the SIMP method [43]:

$$k_e = \rho_e^p k_0$$ (3)

Where $k_0$ denotes the element stiffness matrix for the solid part, and $p$ is the penalty exponent. The BESO method uses the same sensitivity filter as the SIMP method to prevent numerical instabilities in the optimization process. For the stability of the evolutionary process, an average sensitivity method is used as follows:

$$S_{it+1}^j = \frac{S_{it}^j + S_{it-1}^j}{2}$$ (4)

Where $it$ denotes the iteration number, and $j$ is the node number. When the element and node sensitivities are calculated, the design variables are updated using the soft kill method. If the volume constraint is not satisfied, the volume of the next iteration $V_{it+1}$ is calculated from the following equation (5), where $ER$ is the rate of volume evolution:

$$V_{it+1} = V_{it} (1 + ER)$$ (5)

The essential step in optimizing the topology is to identify solid and void areas in the structure. A threshold criterion must be specified to distinguish these areas. In this approach, the boundaries are defined using the contours of higher-dimensional (3D) structural performance, such as strain energy density or Von Mises stress, obtained from finite element analysis in the design space. In structural optimization applications, boundary definition has been done by intersecting the structural performance (SP) distribution with a minimum standard level of performance (MLP). In the present method, the strain energy density is considered as a measure of structural performance. As a result, the following relationship can be defined:

$$\alpha_i = SP_i - MLP$$ (6)

Where $i$ is the node number, and $\alpha$ is the relative performance. The MLP value in each iteration is calculated from the following equation:

$$MLP_{it} = RF_{it} \times \frac{SP_{max}}{VF_{it}}$$ (7)

Where $SP_{max}$ is the maximum performance over the design region, $RF_{it}$ is the redistribution coefficient for the current iteration and $VF_{it}$ the current volume fraction of the solids in the entire design space. According to the above: if $\alpha > 0$, the relevant area can be filled with materials; in the case of $\alpha < 0$, the area should be replaced with ineffective materials (void area); otherwise, it will be the boundary area. The boundary cannot be treated in the same way as the other two areas, as it may lead to undesirable and non-smooth design boundaries. Therefore, in this study, the XFEM is used to deal with boundary elements. To select the correct solid, void, and boundary elements, the following steps are carried out:

- First, the $\alpha$ value is calculated for all nodes of an element. If all nodes are of positive $\alpha$ values, the element is solid, and if none of the nodes has a positive $\alpha$ value, it will be related to the void element. Otherwise, the element is on the border.
Then, the stiffness matrices of solid and void elements are computed using the conventional FEM method. However, for the boundary element, the XFEM is used.

2.2. Extended finite element method

In conventional TO methods, boundary elements are treated as solid or void areas, meaning that the element may be completely solid or empty. Also, due to the square shape of the element, it may cause a non-smooth boundary. Moving beyond the classical FEM, the XFEM was recently developed to represent discontinuities, such as cracks and material-void interfaces, inside finite elements. Besides the polynomial shape functions used for the classical FEM, additional “enrichment” functions are employed in the XFEM to approximate the solution. In this regard, XFEM can be employed in TO problems to handle the material-void discontinuity introduced by the evolving boundary during the optimization process, potentially enabling a sub-element boundary representation. Therefore, in this research, smoothing the structural boundary is considered using the XFEM: first, a point is created in the solid region of a boundary element; then, the point is connected to the nodes with positive $\alpha$ values and intersections of the boundary with the element; finally, the solid region of the boundary element is divided into a few sub-triangles. This process is illustrated in Fig. 1.

![Fig. 1. Boundary element and its sub-triangles](image)

The following linear interpolation relationship is used to calculate the intersection points of the boundary and the element:

$$ x_i = \frac{l_{ij}}{1 - (\alpha_i/\alpha_j)} $$

where $l_{ij}$ is the distance between nodes $i$ and $j$. After converting the area to sub-triangles, the stiffness matrix value of each sub-triangle must be calculated. In this study, the stiffness matrix of a 2D boundary element is defined by:
\[ k_e = \int_{\Omega} B^T D_S B t d\Omega \]  
(9)

where \( \Omega \) is the element domain, \( B \) is the displacement differentiation matrix, \( D_S \) is the elasticity matrix of the solid material, and \( t \) is the element thickness. Equation (9) omits the void sub-domain of the element and performs integration only on its solid sub-domain. In the case of 2D quadrilateral elements, the integral in equation (9) can be numerically calculated by performing the integration using the Gauss quadrature method:

\[ k_e = \sum_{i=1}^{q} \int_{ST_i} B^T D_S B t d\Omega \]  
(10)

where \( q \) is the number of sub-triangles inside the element, and \( ST_i \) denotes the \( i \)-th sub-triangle. By applying the Gauss quadrature method and calculating the integrals over the sub-triangles, the 2D integrals in terms of the physical coordinates are assigned to the triangle’s natural coordinates and represented as a series of weighted functions:

\[ k_e = \sum_{i=1}^{q} \sum_{j=1}^{m} A_j w_j g(\xi_j^1, \xi_j^2, \xi_j^3) \]  
(11)

where \( g = B^T D_S B t \), \( m \) is the number of Gaussian points, \( \xi_j^1, \xi_j^2 \) and \( \xi_j^3 \) are the coordinates of the Gaussian points, \( A \) is the area of a sub-triangle, and \( w \) is the weighting factor.

The XFEM for hole and inclusion suggested by Sukumar et al. [45] can be used to represent the discontinuities of voids using a single-step function

\[ u(x) = \sum_i u_i N_i(x) H(x) \]  
(12)

where \( N_i \) are the classical shape functions related to the nodal degrees of freedom \( u_i \). The unit step function \( H(x) \) can be defined as:

\[ H(x) = \begin{cases} 1 & \text{if } x \in \Omega_S \\ 0 & \text{if } x \notin \Omega_S \end{cases} \]  
(13)

where \( \Omega_S \) is the solid sub-domain. Based on the above definition, the step function for solid regions is considered as 1. In contrast, the value is zero for void elements. In the above formulation (12) for the XFEM, the enrichment of shape functions is not involved; instead, the single-step function is used to define discontinuities. Hence, the same degrees of freedom as those used for the FEM will be defined for the XFEM approximation space of a particular problem. To understand the developed XFEM, as suggested by Sukumar et al. [45], the void part of the element can be removed from the integral in calculating the element stiffness matrix, and the integration is merely performed over the solid sub-domain of boundary elements. The flowchart of the deterministic topology optimization using XFEM is shown in Fig. 2.

2.3. Topology optimization under uncertainty

For random system properties (random excitations, material uncertainty, or production errors), the response \( \mathbf{U} \) becomes a random field, and the objective function is a random variable. The optimization problem under uncertainty is formulated as follows:
\[
\begin{align*}
\min_{\rho} & \quad m_c(\rho) + \kappa \sigma_c(\rho) \\
\text{subject to:} & \quad m_V(\rho)/V^0 = V^c \\
& \quad K(\rho, \zeta)U(\rho, \zeta) = F \\
& \quad 0 \leq \rho \leq 1
\end{align*}
\]  

(14)

where \( \kappa \) is the weighting coefficient, \( \zeta \) is the random variable, and \( m_V \) is the mean of volume. \( m_c(\rho) \) and \( \sigma_c(\rho) \) are mean and standard deviation (STD) of the compliance, respectively, which are defined as follows:

\[
m_c(\rho) = \mathbb{E}\{C(\rho, \zeta)\} \
\]

\[
\sigma_c(\rho) = \sqrt{\mathbb{E}\{C^2(\rho, \zeta)\} - [\mathbb{E}\{C(\rho, \zeta)\}]^2} = \sqrt{\text{Var}(C)} 
\]  

(15)

The sensitivity of the objective function in equation (14) to the design variable \( \rho \) can be calculated as follows:

\[
S = \frac{\partial m_c}{\partial \rho} + \kappa \frac{\partial \sigma_c}{\partial \rho} 
\]  

(16)

2.3.1. Representation of uncertainty

To express uncertainty, a truncated KLE is used to model the random field. This expansion provides a mapping of a relatively small number of independent random variables to the types of random domains that are common to many physical processes. The random field \( Z \) as a result of expansion does not have the optimal range (between zero and one) for topology optimization. Hence, an inverse conversion sampling on \( Z \) is used to ensure the desired variation of the random variables.

Fig. 2. The flowchart of the deterministic topology optimization using XFEM
The square exponential correlation function has been used to define the KLE of $Z$ \cite{20}, i.e.

$$R_{yy'} = \exp\left(-\frac{d^2}{2l_c^2}\right)$$  \hfill (17)

where $d = |y - y'|$ is the distance between the centres of two elements, and $l_c$ is the length of the correlation. $n$ Eigenvalue-eigenvector pairs of the correlation matrix, i.e. $(\lambda_i, y_i)$, are used to generate the expansion analysis and create the following equation (18) \cite{20}:

$$Z(\rho, \zeta) = \sum_{i=1}^{n} \sqrt{\lambda_i} Y_i(\rho) \varphi_i(\zeta) \approx \sum_{i=1}^{n_{mode}} \sqrt{\lambda_i} Y_i(\rho) \varphi_i(\zeta)$$  \hfill (18)

Where $\varphi_i$ is a random variable. Expansion to the first $n_{mode} < n$ modes is used to reduce the dimensions. Note that the number of $n_{mode}$ depends on the length of the correlation $l_c$ and the type of correlation function. Stochastic variables are assumed to have a uniform distribution with zero mean and unit variance, i.e. $\varphi_i(\zeta) = \zeta_i = U[-\sqrt{3}, \sqrt{3}]$.

Since the realization of the KLE from the $Z$ field is not in the range of zero and one ($Z(\rho, \zeta) \notin [0, 1]$), an operation shall be performed on this field to transform the range to [0, 1]. For this purpose, the cumulative density function (CDF) is applied to the random field $Z$, and a new random field $\hat{Z}(\rho, \zeta) \in [0, 1]$ is created \cite{20}:

$$\hat{Z}(\rho, \zeta) = CDF(Z(\rho, \zeta))$$  \hfill (19)

2.3.2. Propagation of uncertainty

One of the popular methods used for dimension reduction and uncertainty propagation is the sparse grid method, which is based on sparse tensor products. The construction of this method is shown as follows. Suppose that $Q_i^{(1)} f$ is a family of quadrature rules, $\Delta_i^{(1)} f$ is also a quadrature rule \cite{12}:

$$\Delta_i^{(1)} f \equiv \left(Q_i^{(1)} f - Q_{i-1}^{(1)} f\right)$$

$$Q_0^{(1)} f \equiv 0$$  \hfill (20)

For nested formulas, $\Delta_i^{(1)} f$ contains the set of nodes $Q_i^{(1)} f$ with weights equal to the difference of weights between levels $l$ and $(l - 1)$. By introducing the multi-index $I = (l_1, \ldots, l_N) \in \mathbb{N}^N$, the sparse cubature can be constructed as:

$$|I| \equiv \sum_{i=1}^{N} l_i$$  \hfill (21)

At level $l$, this multi-index is applied, and the sparse cubature formula is represented by:

$$Q_l^{(N)} f \equiv \sum_{|I|=l+1+\cdots+N-1} \left(\Delta_{l_1}^{(1)} \otimes \cdots \otimes \Delta_{l_N}^{(1)} \right) f$$  \hfill (22)

where the multi-index expression of the support nodes is $|I|$, $|I| = l_1 + \cdots + l_d$. $N$ is the dimension of objective function $f$. Using a recursive manner, the interpolant can be expressed as follows:
\[ Q_l^{(d)} f = \sum_{l+1 \leq |\mathbf{k}+d|} (-1)^{l+d-|\mathbf{k}|} \left( \frac{d-1}{l+d-|\mathbf{k}|} \right) \left( Q_{k_1}^{(1)} \otimes \cdots \otimes Q_{k_d}^{(1)} \right) f \]  

(23)

The weight \( w_i \) corresponding to the \( i \)th collocation point \( \zeta_i \) is defined with the Smolyak algorithm:

\[ w_i = (-1)^{l+d-|\mathbf{k}|} \left( \frac{d-1}{l+d-|\mathbf{k}|} \right) \left( w_{k_1}^{i_1} \otimes \cdots \otimes w_{k_d}^{i_d} \right) \]  

(24)

Then, the mean and standard deviation of the function \( f \) by sparse grid method can be computed by:

\[ \mathbb{E}[f] = \sum_k w_k f(Z_k) \]  

\[ \text{Var}[f] = \mathbb{E}\left[f^2\right] - \left(\mathbb{E}[f]\right)^2 \]  

(25)

The sparse grid method discretizes the stochastic space in a hierarchical structure based on nested collocation nodes, such as the Chebyshev or Gauss-Patterson nodes, leading to Clenshaw-Curtis or Gauss-Patterson rule, respectively. Here, the sparse grid of Clenshaw-Curtis type with non-equidistant nodes is used for the calculation of weight \( w_i \) and collocation point \( \zeta_k \).

2.3.3. Modeling of material uncertainty

In many applications of mechanical systems, the uncertainty of material properties such as elastic modulus or thermal conductivity is modelled by the Gaussian random field. It should be emphasized that both the modulus of elasticity and thermal conductivity have positive values. The negative values assigned to them by the Gaussian assumption have no physical meaning. The elastic modulus can be modelled as follows:

\[ E = h[Z(\rho, \zeta)] \]  

(26)

Where \( h[] \) is a derivative function of a real value and \( [\cdot] \) is a cumulative distribution function (CDF). Relatively simple and physically acceptable margin distributions for modelling the material properties of systems include log-normal, beta, or uniform distribution. According to the stated cases, the material property can be expressed as equation (27) for uncertainty:

\[ E = \rho^p E_m \]

\[ E_m = \beta_1 Z(\rho, \zeta) + \beta_2 \]  

(27)

where \( \beta_1 \) and \( \beta_2 \) are control parameters that confirm the presence of the function \( Z(\rho, \zeta) \) in the range of zero and one.

2.3.4. Modeling of loading uncertainty

In this subsection, uncertainty in applying load for linear elastic structures has been investigated. Uncertainty has been mentioned in the angle and amplitude of load. Two independent random variables determine these uncertainties. An arbitrary angle is assumed to have a uniform distribution. Therefore, this random field is represented by equation (26). For size, the uncertainty is shown by a Gaussian distribution with a mean of 1 and an STD of 0.3. So we will have:

\[ f(\rho, \zeta) = FA(\rho, \zeta) \]

\[ F = N(1,0.3), \quad A(\rho, \zeta) = \sum_{l=1}^{n_{mod,e}} \sqrt{\lambda_l}Y_l(\rho)\phi_l(\zeta) \]  

(27)
Where $F$ is the load amplitude, and $A$ is the load angle.

Finally, the sparse grid collocation section connects with the presented method as follows:

Define random field

Define uncertainty

In Material or Loading

Applying this uncertainty to calculate objective and sensitivity by using sparse grid collocation

Solve optimization by using XFEM

3. Numerical Examples

All of the numerical examples are run and analyzed in MATLAB. In Fig. 3, the first 100 eigenvalues of the correlation matrix $R$ have been shown, where their fast modal decay is apparent. In practice, firstly truncate the KLE up to $n$ terms and then use the ratio \( \frac{\sum_{i=1}^{n_{\text{mode}}^n} \lambda_i}{\sum_{i=1}^{n} \lambda_i} \) to check the sufficiency of the truncated modes, which indicate the first $n_{\text{mode}}$ modes to represent the random field. This measure for $n_{\text{mode}} = 4$ is 0.9566, i.e. this truncation yields a 96% $Z$ representation which deems sufficient. Note that in the numerical examples, the method presented is referred to as the ES-XFEM. The SIMP method also refers to the topology optimization method under uncertainty, which has used the SIMP method and classical FEM in its deterministic topology optimization problem [24-26].

3.1. 2D MBB Beam (loading uncertainty)

The design range, boundaries, and loading conditions of a two-dimensional beam with a simply supported constraint are shown in Fig. 4. This problem is a typical benchmark example that has been widely used for methodology validation in robust topology optimization [11,12,24]. Therefore, it is beneficial to adopt this example for validating the proposed ES-XFEM method. The dimensions of the beam are considered as $L = 90$ mm and $H = 30$ mm, and its thickness is equal to 1 mm. It is assumed that the material has a Young modulus of 1 MPa for the solid phase.
and a Poisson ratio of 0.3. A random load is considered, in which two independent random variables determine its angle and amplitude. It is assumed that the applied load angle follows a uniform distribution in the distance $[\pi/4, 3\pi/4]$. The force magnitude of the Gaussian distribution has an average of 1 and an STD of 0.3.

![Fig. 4. Design domain and boundary condition of a 2D MBB beam](image)

The goal of robust optimization is to minimize the softness of the structure, which means optimizing the mean and minimizing the STD under the volume constraint. In this example, weight $\kappa$ is equal to one. For structure analysis, quadrilateral elements are used to discretize the design domain. The final volume is regarded as 0.4 of the initial volume. The optimal topologies obtained by the ES-XFEM developed in two modes (deterministic and robust TO) have been shown in Fig. 5.

![Fig. 5. Topology optimization using the ES-XFEM method: (a) Under uncertainty, (b) Deterministic.](image)

As shown in Fig. 5, the results of deterministic and robust optimizations have different topologies. A significant difference in the robust design represents an asymmetric design compared to the deterministic design characterized by a seemingly symmetrical configuration. This can be attributed to the fact that the right support of the beam can move horizontally in the deterministic design. Because the force exerted on this point is very small, its displacement can also be neglected. For this reason, the final topology has an almost symmetrical appearance. However, in a robust design, the applied force also has a horizontal component. As a result, the left support of the beam is effected by the resulting deformation, and the asymmetry in the structure is clearly visible. Fig. 6 and Table 1 present a comparison between the ES-XFEM and SIMP methods using the classical FEM. As can be seen, the final structure has smoother boundaries for the ES-XFEM, as compared to that by the SIMP. Since the main objectives in the RTO are to obtain the optimal conditions for the mean data and smallest standard deviation, Table 1 shows that these objectives are achieved in the presented ES-XFEM. Therefore, a more stable structure is obtained.

The optimal topologies obtained by different RTO methods are compared in Table 2. It can be found that the addition of the XFEM to the evolutionary structural topology optimization process works better than other methods in terms of robustness and cost in robust design. Finally, by
comparing the STD of these methods, the superiority of the ES-XFEM over other methods can be confirmed again because, in topology optimization, the ultimate goal is to optimize the mean and minimize the STD for stable design topology. Also, based on the final topology in Fig. 6, a smooth boundary has been created, and there is no need for any post-processing (re-meshing) techniques due to the adopted XFEM for boundary representation. On the contrary, the topology obtained by the SIMP method has a non-smooth boundary and thus needs additional post-processing techniques to have a smooth boundary and robust topology.

![Fig. 6. Robust topology optimization was obtained by: (a) ES-XFEM and (b) SIMP.](image)

<table>
<thead>
<tr>
<th>Method</th>
<th>Mean</th>
<th>STD</th>
<th>Compliance</th>
<th>Time (s)</th>
</tr>
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<td>SIMP</td>
<td>19.53</td>
<td>8.59</td>
<td>28.12</td>
<td>55</td>
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<tr>
<td>ES-XFEM</td>
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<td>7.57</td>
<td>26.74</td>
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</tr>
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</table>

**Table 1. Comparison of robust design by two methods**

**Table 2. Comparison of results obtained from different RTO methods**

<table>
<thead>
<tr>
<th>Method</th>
<th>Mean</th>
<th>STD</th>
<th>Compliance</th>
</tr>
</thead>
<tbody>
<tr>
<td>ES-XFEM</td>
<td>19.17</td>
<td>7.57</td>
<td>26.74</td>
</tr>
<tr>
<td>SIMP</td>
<td>19.53</td>
<td>8.59</td>
<td>28.12</td>
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<tr>
<td>Monte Carlo [16]</td>
<td>21.37</td>
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<td>Tensor product grid (l = 3) [16]</td>
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<td>11.34</td>
<td>33.11</td>
</tr>
<tr>
<td>Sparse grid (l = 3) [16]</td>
<td>21.52</td>
<td>11.27</td>
<td>32.79</td>
</tr>
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</table>

**3.2. 2D Cantilever Beam (Material uncertainty)**

The design range geometry, boundary conditions, and applied load are shown in Fig. 7. The length, height, and thickness of the slope are \(L, 5L/8\) and 1, respectively. The control parameters are considered as \(b_1 = 0.6\) and \(b_2 = 0.5\). The final volume is considered 0.25 of the initial volume. It is assumed that the substance has a Poisson ratio of 0.3. The results are obtained with the penalization parameter \(p = 3\). The sample under study has uncertainty in Young’s modulus. It is assumed that Young’s modulus for the solid part has a uniform distribution in the range of [0.1, 1.9]. The simulation results for the ES-XFEM, SIMP, PCE, and MC methods are presented in Table 3. It can be seen that the results obtained using the ES-XFEM are better than the results by other methods. In other words, the final topology obtained using the ES-XFEM is more robust. Its minimum mean and lower STD make the final topology less sensitive than that by other methods.
Fig. 7. Boundary condition and geometry of a cantilever plate

Table 3. Mean and STD of different methods (κ=1)

<table>
<thead>
<tr>
<th>Method</th>
<th>Mean</th>
<th>STD</th>
<th>Mean</th>
<th>STD</th>
<th>Mean</th>
<th>STD</th>
<th>Mean</th>
<th>STD</th>
</tr>
</thead>
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<tr>
<td>ES-XFEM</td>
<td>0.802</td>
<td>0.058</td>
<td>0.882</td>
<td>0.087</td>
<td>0.885</td>
<td>0.115</td>
<td>0.884</td>
<td>0.115</td>
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<td>SIMP</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PCE [19]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Monte Carlo</td>
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<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>(500 Points) [19]</td>
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<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

The iteration over the objective values in equation (14) for this example is depicted in Fig. 8. It is observed that as the number of iterations increases, the objective values change from 0.44 to 0.86 and converges when the volume fraction conditions are determined, and the difference criterion of the objective function in two consecutive iterations is met.

Fig. 8. Objective values versus number of iterations

Fig. 9 shows the topology obtained using the XFEM in a deterministic state and under uncertainty. It can be seen that the final topology in the uncertain design does not have symmetric distribution. The reason is related to the stochastic distribution of material. Also, Fig. 10 and Table 4 show topology optimization results with/without the XFEM for κ=1. As seen from the results, the use of XFEM in the optimization process will create a robust structure, which has a minor mean and the least STD. Also, the results show that in particular mesh sizes, the present method has a smoother boundary region and better accuracy than the SIMP method. This means that the
The final structural topology is more stable. The main reason for this behaviour is the correct treatment of boundary elements, which results in having an optimal topology with a smoother boundary.

**Fig. 9.** Optimized topology obtained by: (a) Deterministic Method and (b) Robust Method.

**Table 4.** Result comparison of different robust designs

<table>
<thead>
<tr>
<th>Method</th>
<th>Mean</th>
<th>STD</th>
<th>Compliance</th>
<th>Time (S)</th>
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<tr>
<td>SIMP</td>
<td>0.882</td>
<td>0.087</td>
<td>0.969</td>
<td>104</td>
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<td>0.802</td>
<td>0.058</td>
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**4. Conclusions**

In this research, a method for optimizing the topology under uncertainty is introduced, which relies on sparse grid collocation methods to propagate the uncertainty of the objective function and constraints. Also, an XFEM for treatment with boundary elements is introduced. Different numerical examples are considered to optimize the topology under uncertainty. The results show that the use of the XFEM for dealing with boundary elements has significant advantages. Not only does it not use time-consuming post-processing techniques, but also produces a structure with a smooth boundary that does not need to be further interpreted or processed. Comparison of the mean and STD obtained by methods under different uncertainties shows that the reduction of STD
is higher than that of the mean of compliance using the method based on the XFEM. This means that the final topology is more stable and less sensitive using the XFEM, which can be confirmed from the more systematic optimal topology obtained with this method.

Also, it is observed that the amount of changes in time is much greater than the amount of improvement in compliance when the XFEM is used. Of course, one of the challenges in constructing samples, for example, by additive manufacturing, obtained from TO is the interpretation and smoothing of the structural boundary. To solve this problem, TO results must be interpreted by using another method, which may take longer to solve than the XFEM. In this regard, this increase in solution time caused by XFEM in this study may not be a concern. For future works, the presented method will be extended to real processes and non-linear structures with more complex analyses.

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**Compliance with ethical standards**

**Conflict of interest** The authors declare that they have no conflict of interest.

**Abbreviations**

<table>
<thead>
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<th>words or phrase</th>
<th>Symbol</th>
<th>words or phrase</th>
<th>Symbol</th>
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<td>Topology Optimization</td>
<td>TO</td>
<td>Robust Topology Optimization</td>
<td>RTO</td>
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<td>FE</td>
<td>eXtended Finite Element Method</td>
<td>XFEM</td>
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<td>ES-XFEM</td>
<td>Loading Uncertainty</td>
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<td>Geometry Uncertainty</td>
<td>GU</td>
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<td>HU</td>
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<td>RBTO</td>
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<td>TE</td>
<td>Stochastic Galerkin</td>
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<td>LSM</td>
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<td>KLE</td>
<td>Gaussian Multivariate Quadratic Function</td>
<td>GMQF</td>
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<td>Sparse Grid method</td>
<td>SpGM</td>
<td>horizontal axis wind turbine</td>
<td>HAWT</td>
</tr>
<tr>
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<td>BESO</td>
<td>Solid Isotropic Material with Penalisation</td>
<td>SIMP</td>
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**References**


