Revisit of the Variable Stiffness Method for Aeroelastic Computations with/without Thermal Effects Based on Computational Fluid Dynamics

Enqian Quan\textsuperscript{a}, Min Xu\textsuperscript{a}, Weigang Yao\textsuperscript{b,∗}, Xunliang Yan\textsuperscript{a}

\textsuperscript{a}Northwestern Polytechnical University, Xi’an 710072, China
\textsuperscript{b}Faculty of Computing, Engineering and Media, De Montfort University, Leicester LE1 9BH, UK

\textbf{Abstract}

In the present study, a variable stiffness method (VSM) is revisited for aeroelastic computations with/without thermal effects. Unlike the traditional CFD/CSM coupling method (TM), which predicts aeroelastic responses by varying freestream conditions (e.g. freestream density, velocity), the freestream conditions in VSM can be fixed. The VSM is first verified theoretically and adopted to predict nonlinear aeroelastic responses. For aeroelastic computations, the method is applied to predict flutter onset of Isogai wing section and limit cycle oscillation (LCO) of Goland\textsuperscript{+} wing at transonic conditions. The structural free-play nonlinearity is included in the Isogai wing section aeroelastic system to further demonstrate the method. The aeroelastic computation with thermal effects is considered as the flutter onset prediction of a simply supported panel in supersonic flow. It is shown that the VSM method can replicate the nonlinear aeroelastic responses predicted by its traditional

\textsuperscript{∗}Corresponding author

\textit{Email address: weigang.yao@dmu.ac.uk} (Weigang Yao)
CFD/CSM coupling method counterpart under the same flow similarity parameters (e.g. Mach number, Reynolds number), whereby the freestream conditions need to be adjusted. Limitations of the VSM are also pointed out and discussed. To further assess the VSM, an ARMA model is constructed under the framework of the VSM and demonstrated by flutter onset prediction of the Isogai wing section at transonic regime.

**Keywords:** thermal effects, Aeroelasticity, Reduced-order model

---

**Nomenclature**

- $\alpha$  angle of attack, $rad$
- $\alpha_f$  initial free-play magnitude of pitch motion, $deg$
- $\alpha_m, \alpha_0$  mean angle of attack and amplitude of pitching motion, $deg$
- $\alpha_R, \beta_R$  Rayleigh damping constants
- $\alpha_s$  thermal expansion coefficients, $1/K$
- $\beta$  transform coefficient
- $\varepsilon_0$  thermal strain
- $A_i, B_i$  constant coefficients matrices
- $a$  structural nodal displacement vector, $m$
- $A(a)$  characteristic matrix
\( b^* \) body force and damping force

\( B \) transform matrix

\( C, \overline{C} \) dimensional and nondimensional total damping matrix

\( ds \) outer normal vector to the boundary

\( D, \overline{D} \) dimensional and nondimensional elastic moduli tensor

\( E, E^e \) Green-Lagrange and compatible Green-Lagrange strain tensor

\( F_c, F_v \) convective and viscous flux vectors

\( f \) aerodynamic force coefficient vector

\( I \) identity matrix

\( K(a) \) total stiffness matrix

\( L(a), \overline{L}(a) \) dimensional and nondimensional eigenvalue matrix

\( M \) total mass matrix

\( P \) total load vector

\( P_a \) external force matrix

\( P_T \) thermal load matrix

\( Q \) conservative flow variables

\( Q(a) \) eigenvector matrix

\( S \) the second Piola-Kirchhoff stress tensor, Pa
traction vector

\( \mathbf{u}^*, \mathbf{t}^* \) prescribed displacement and traction

\( \mathbf{U}_g \) velocity of the moving domain

\( \mathbf{U} \) \([u, v, w]\), flow velocity in a Cartesian system

\( \mathbf{u} \) displacement vector, \( m \)

\( \mathbf{W}_s \) strain energy density

\( \mathbf{y} \) system output vector

\( \Delta T \) temperature differential, \( K \)

\( \Delta T_{cr} \) critical temperature differential, \( K \)

\( \delta_{ij} \) Kronecker delta

\( \dot{q}_i \) heat flux tensor

\( \gamma \) ratio of specific heats

\( \lambda \) \( 2qa^3/D \), nondimensional dynamic pressure

\( \mu \) mass ratio

\( \mu, \mu^l, \mu^T \) total, laminar and turbulence viscosity, respectively.

\( \nu \) Poisson’s ratio

\( \omega \) angular velocity, \( rad/s \)

\( \Omega, \partial\Omega \) control volume and boundary of control volume
\[ \omega_0 = \pi^2 \sqrt{D/(\rho_s h_s a_s^4)}, \text{ reference circular frequency, } rad/s \]

\[ \Omega_0, \partial \Omega_0 \text{ structural control domain and boundary} \]

\[ \omega_d \text{ adjustable circular frequency, } rad/s \]

\[ \omega_h, \omega_\alpha \text{ uncoupled natural frequencies of airfoil in plunging and pitching, respectively, } rad/s \]

\[ \omega_i \text{ } i^{th}\text{-order circular natural frequency, } rad/s \]

\[ \rho \text{ air density, } kg/m^3 \]

\[ \rho_s \text{ structural mass density, } kg/m^3 \]

\[ \tau \text{ nondimensional time} \]

\[ \tau_0 \text{ thickness-to-chord ratio} \]

\[ \tau_{ij} \text{ shear stress tensor} \]

\[ \xi_i \text{ } i^{th}\text{-order generalized structural displacement} \]

\[ \zeta \text{ damping ratio} \]

\[ a \text{ nondimensional location of elastic axis} \]

\[ a_s \text{ plate length, } m \]

\[ b, c \text{ semichord and chord length, } m \]

\[ c_p \text{ specific heat at constant pressure, } J/(kg \cdot K) \]

\[ C_l \text{ lift coefficient} \]
\( C_m \)  moment coefficient about elastic axis

\( D = \frac{E_s h_s^3}{12(1 - \nu^2)} \), plate stiffness

\( E \)  total energy

\( E_s \)  Young's modulus, Pa

\( h_s \)  plate thickness, m

\( k_c \)  reduced frequency

\( K_f = \frac{\omega}{\omega_0} \), nondimensional frequency

\( K_h, K_\alpha \) plunging and pitching spring constant

\( Ma \)  Mach number

\( n_a, n_b \) input and output delay

\( P \)  pressure, Pa

\( Pr \)  Prandtl number

\( Q \)  heat flux density, W/m\(^2\)

\( q = \rho_\infty U_\infty^2 / 2 \) freestream dynamic pressure, Pa

\( R \)  ideal gas constant, J/(kg \cdot K)

\( r_\alpha \)  radius of gyration about elastic axis

\( Re \)  Reynolds number

\( S_\sigma \)  prescribed traction boundary
\( S_a \)  prescribed displacement boundary

\( T \)  temperature, \( K \)

\( \tau_{\text{physics}} \)  physical time, \( s \)

\( V_f \)  nondimensional velocity

\( x_{\alpha} \)  static unbalance

\( x_m \)  pitching axis (\% chord root)

Superscripts

\( c \)  current value

\( e \)  within a typical element

\( \text{ref} \)  reference value

Subscripts

\( \infty \)  freestream value

\( c \)  current value

\( e \)  within a typical element

\( \text{ref} \)  reference value
1. Introduction

Aeroelasticity phenomena arise due to strong interactions between the flexible aircraft and the surrounding flow. These interactions are nonlinear and causes complex aeroelastic response such as flutter, limit cycle oscillation and chaotic motion. The development of computational methods for aeroelastic analysis with efficiency receives continuous interests. The aeroelastic system can be further complicated in hypersonic flow, whereby aerodynamic heating alters the structure significantly due to material degradation and thermal stress. Therefore, it is of particular interest to consider thermal effect in aeroelastic analysis for hypersonic aircraft design.

Substantial work has been done to develop numerical methods for aeroelastic computations with or without thermal effects. Isogai [19, 20, 21] investigated the transonic flutter characteristics of a NACA64A010 airfoil with pitch and plunge degrees of freedom by using the transonic small perturbation (TSP) method, whereby a sharp decrease is observed in the flutter speed at transonic regime. The advances in computer hardware and numerical algorithm render computational fluid dynamics (CFD) techniques feasible for aircraft design, such as aerodynamic modeling, aeroelastic computation and aircraft shape optimization, etc.[12]. Recently, the coupling of high fidelity CFD and computational structural mechanics (CSM) has attracted considerable interest for aeroelastic computations [14]. To our best knowledge, Rausch et al. [32] pioneered the CFD methodology in transonic aeroelastic analysis, where an Euler CFD solver is adopted for flutter onset prediction of the AGARD 445.6 wing and a supersonic transport aircraft. The results agree with the experimental data reasonabllly well, showing that CFD is a
reliable alternative for aeroelastic analysis at transonic regime. Similarly, Alonso and Jameson [1] computed flutter onset boundary of the Isogai wing by using CFD/CSM coupling and a good agreement is found with the literature. In addition to aerodynamic nonlinearity, structural nonlinearity also significantly affects the characteristics of the aeroelastic system. Kim and Lee [23] adopted a two-dimensional unsteady Euler solver for aeroelastic computation, whereby free-play nonlinearity is considered. In [15], the panel flutter onset is predicted by solving the Navier-Stokes/Euler equations and the von Kármán plate theory is used to construct the panel structural model. For more practical engineering problems, Mian et al. [31] applied the CFD/CSM coupling method to perform nonlinear static aeroelastic analysis of a high-aspect-ratio wing with geometric nonlinearity.

Aerothermoelasticity remains an attractive topic in high-speed aircraft design as the aerodynamic heating is more pronounced in supersonic flow. Dowell [10] employed Von Kármán’s large deflection plate theory and quasi-steady aerodynamic theory to investigate the panel flutter problem subject to uniform temperature differential distribution. Xue and Mei [42, 41] further investigated the problem with non-uniform temperature effects for two-dimensional and three-dimensional isotropic panels of arbitrary shape using the discrete Kirchhoff theory (DKT) triangular plate element. Culler and McNamara [7] developed an aerothermoelastic model to investigate the influence of transient temperature distribution and material property degradation. However, only a low fidelity method is employed to compute the aerodynamic forces and aerodynamic heating, which is considered to be inadequate to capture aerodynamic nonlinearity. A time-adaptive multi-physics coupling
strategy is proposed by Chen et al. [4] for aerothermoelastic computation. In the work, a typical low-aspect-ratio hypersonic wing is used to study the impact of sustained aerothermodynamic load on the hypersonic wing structure. The results show that the proposed method is reliable, applicable and efficient for aerothermoelastic analysis.

High fidelity model, such as CFD and CSM, remains the state-of-art for aeroelastic and aerothermoelastic computation, however, the high computation cost renders the methods less attractive for industry routine analysis. Therefore, it is desirable to develop fast, accurate and ease-of-use methods to accelerate otherwise impractical and intractable simulation in engineering application. Reduced Order Modelling (ROM) technique approximates the original high-fidelity systems or full order model (FOM) with a low-order model, which retain the most significant dynamics of the FOM and offers an order of magnitude efficient improvement. The area of ROM construction remains an active research topic in aeroelasticity [29]. ROM construction techniques can be categorized into two main approaches, namely, system identification and Galerkin projection. As far as the authors are aware, Volterra series is first adopted by Silva [36] for flutter onset computation, whereby only impulse response of the CFD system is required to construct a linear ROM. Recently, Yang et al. [43] developed a novel ROM for nonlinear unsteady aerodynamic computation by including nonlinear terms in the linear state space ROM identified by eigen system realization algorithm (ERA). Proper orthogonal decomposition (POD) has been widely used to extract reduced order basis or mode vector for ROM construction based on Galerkin projection [16, 11]. Yao and Marques [44] proposed a novel ROM
using the Discrete Empirical Interpolation Method (DEIM) for nonlinear aerodynamic and aeroelastic computation. RBF artificial neural network (ANN) is adopted to build a nonlinear mapping function between structural displacements and flow variables of interest on the selected interpolation points identified by DEIM. The flow field is then reconstructed in the sense of least-square approximation by the reduced basis extracted by POD. Results suggest that the ROM is able to replicate nonlinear aerodynamic force and reconstruct the flow field of interest with sufficient accuracy at a fraction of the cost of the FOM. Huang and Friedmann [18] enhanced their aerothermoelastic computational framework by integrating a POD-kriging based ROM and an efficient coupling scheme, which reduces computational time by four orders of magnitude approximately.

As aforementioned, CFD/CSM coupling method is able to capture nonlinearity of both aerodynamic and structural systems and remains the state of art for aeroelastic and aerothermoelastic computation. However, following the concept of traditional CFD/CSM coupling method, which is abbreviated as "TM" in this paper, the frequency and amplitude of the coupling system are computed by varying freestream conditions such as freestream velocity and density, which needs to interrogate the steady CFD solver and quickly becomes prohibitive for problems with large number of aerodynamic grid points. Instead of adjusting freestream conditions, the method called variable stiffness method (VSM), which computes the flutter onset and nonlinear LCO responses by varying the natural frequency (structural stiffness) of structural system, is investigated in the present study. Although this method has been used in the previous research [47, 17], there is still a lack
of systematic investigation in the aeroelastic and aerothermoelastic community. In the present study, the validity of the VSM method is first verified by theoretical derivation for aeroelastic system considering thermal effect and demonstrated by three representative aeroelastic systems, namely Isogai and Goland+ wing models and a simply supported panel at a supersonic Mach number \((Ma = 1.2)\). The limitation of the VSM method is also discussed through theoretical analysis and numerical simulation and further investigation is suggested for future exploration.

The paper is organized as follows. Section 2 introduces the formulation of fluid and structural system. Theoretical derivation for the VSM is described in Section 2.3. Results and comparison are provided in Section 3. Section 3.5 discusses the link between the VSM and reduced-order models. Finally, main conclusions are drawn in Section 4.

2. Numerical Methodology

For the sake of completeness, the flow governing equations of the CFD system and the implementation of the numerical schemes used for the CFD solver are described. Later, the general formulation of structural model is presented in Section 2.2.

2.1. Reynolds-Averaged Navier–Stokes Equations (RANS)

Consider a domain \(\Omega \in \mathbb{R}^3\), compressible unsteady Navier-Stokes equations can be expressed in an arbitrary Lagrangian-Eulerian (ALE) form as

\[
\frac{\partial}{\partial t} \int_{\Omega} Q d\Omega + \int_{\partial\Omega} (F_c - F_v) \cdot ds = 0,
\]  

(1)
where the conservative variables are given by \( Q = [\rho, \rho \mathbf{U}, \rho E]^{T} \), and \( \mathbf{U} = [u, v, w] \) is the flow speed in a Cartesian system of reference. The convective and viscous fluxes are defined as

\[
\mathbf{F}_{c} = \begin{bmatrix}
\rho(\mathbf{U} - \mathbf{U}_{g}) \\
\rho \mathbf{U} \otimes (\mathbf{U} - \mathbf{U}_{g}) + p \mathbf{I} \\
\rho E(\mathbf{U} - \mathbf{U}_{g}) + p \mathbf{U}
\end{bmatrix}
\]

\[
\mathbf{F}_{v} = \begin{bmatrix}
0 \\
\tau_{ij} \\
\tau_{ij} u_{j} + \dot{q}_{i}
\end{bmatrix},
\]

where \( \mathbf{U}_{g} \) is the velocity of the moving domain \( \Omega \), and \( \mathbf{I} \) is a \( 3 \times 3 \) identity matrix. The shear stress \( \tau_{ij} \) and heat flux \( \dot{q}_{i} \) are expressed in tensor notation as

\[
\tau_{ij} = \mu \left( \frac{\partial u_{i}}{\partial x_{j}} + \frac{\partial u_{j}}{\partial x_{i}} - \frac{2}{3} \frac{\partial u_{k}}{\partial x_{k}} \delta_{ij} \right),
\]

\[
\dot{q}_{i} = \frac{\mu c_{p}}{Pr} \frac{\partial T}{\partial x_{i}},
\]

where \( c_{p} = \gamma R / (\gamma - 1) \) for a calorically perfect gas, \( \gamma = 1.4 \) for air at standard conditions and \( Pr \) is Prandtl number. In accord with Boussinesq approximation, the total viscosity \( \mu \) is composed of laminar viscosity \( \mu^{l} \) and turbulence viscosity \( \mu^{T} \) as

\[
\mu = \mu^{l} + \mu^{T},
\]

where \( \mu^{l} \) assumes to satisfy Sutherland’s law and \( \mu^{T} \) is obtained by a turbulence model.

In the present study, the flow analysis is performed using an in-house structured-grid, cell-centered, upwind-biased, RANS code. The advection upstream splitting method by pressure-based weight functions (AUSMPW)+ scheme proposed by Kim et al. [25, 24] is adopted to evaluate the inviscid flux and the second order accuracy is achieved with the Monotonic Upstream-Centered Scheme for Conservation Laws (MUSCL) [27] interpolation and the
Van Albada limiter. The turbulence viscosity $\mu^T$ is computed by Menter $k-\omega$ SST turbulence model [30]. Geometric conservation law (GCL) [37] is enforced when CFD/CSM coupling is performed. Lower-upper symmetric-gauss-seidel (LU-SGS) implicit time-stepping scheme [45] is adopted for time advancement. To improve the accuracy of temporal discretization, the second order dual-time-step method [22] is employed to solve unsteady flow solution.

The slip-wall boundary condition is imposed on the solid wall when solving Euler equations and no-slip wall boundary is applied for RANS equations. Far-field boundary is assigned as a non-reflective boundary condition based on one dimensional Riemann invariants. radial basis function (RBF) interpolation [9] is used for mesh deformation in this study, where Wendland’s C2 function is chosen as the basis function. Furthermore, a greedy algorithm is adopted to reduce the number of interpolation points on the solid wall surface [33, 34].

2.2. Formulation of structural model

To investigate the VSM for aeroelastic computations with thermal effects, a variational structure finite element formulation based on Lagrangian framework is presented by considering the Saint-Venant-Kirchhoff material and geometric nonlinearity in this section. The three-field Fraeijs de Veubeke-Hu-Washizu (FHW) variational principle is given by

$$
\Pi (\mathbf{u}, \mathbf{E}, \mathbf{S}) = \int_{\Omega_0} W_s(\mathbf{E})dV + \int_{\Omega_0} \mathbf{S} : [\mathbf{E}^e(\mathbf{u}) - \mathbf{E}] dV - \int_{\Omega_0} \mathbf{u} \cdot (\mathbf{b}^* - \ddot{\mathbf{u}}) \rho_s dV \\
+ \int_{S_o} (\mathbf{u}^* - \mathbf{u}) \cdot \mathbf{t}(\mathbf{S})dS - \int_{S_o} \mathbf{u} \cdot \mathbf{t}^* dS
$$

(5)
where \( \rho_s \) represents the density of material, the Green-Lagrange strain \( \mathbf{E} \), the displacement field \( \mathbf{u} \) and the second Piola-Kirchhoff stress tensor \( \mathbf{S} \) are the independent variables. \( \mathbf{E}^c \) represents the (compatible) Green-Lagrange strain tensor, \( W_s \) is the store strain energy, \( \mathbf{u}^* \) and \( \mathbf{t}^* \) denote the prescribed displacement on the boundary \( S_a \) and the prescribed traction on the boundary \( S_\sigma \), respectively. All the aforementioned variables are defined in the initial configuration \( \Omega_0 \), with \( S_a \cup S_\sigma = \partial \Omega_0 \). In the present study, the body force included in \( \mathbf{b}^* \) is neglected and the traction vector \( \mathbf{t} \) is set to zero since there is no prescribed traction condition. The thermal stress created by the change in temperature is included in \( \mathbf{t}^* \). Detailed expressions of the variables in eq. (5) can be found in [5, 38, 39].

The variational approach leads to the structural dynamic equations in matrix form as the following

\[
M \ddot{\mathbf{a}} + C \dot{\mathbf{a}} + K(a) \mathbf{a} = \mathbf{P}
\]  

where \( \mathbf{a} \) is the structural nodal displacement vector, \( M \), \( C \), \( K(a) \), \( \mathbf{P} \) are the global mass, damping, stiffness matrix and load vector, respectively.

For problems when temperature differential is considered, the load vector in the element \( \mathbf{P}^e \) is defined as

\[
\mathbf{P}^e = \mathbf{P}_a^e + \mathbf{P}_T^e = \mathbf{P}_a^e + \int_{\Omega_e} \mathbf{B}^T \mathbf{D} \mathbf{\varepsilon}_0 d\Omega
\]

\[
\mathbf{\varepsilon}_0 = \alpha_s \Delta T [1 \ 1 \ 1 \ 0 \ 0 \ 0]^T
\]

where \( \mathbf{P}_a^e \) is the external force vector, whereas \( \mathbf{P}_T^e \) denotes the thermal load vector, \( \mathbf{B} \) the deformation-dependent transform matrix, and \( \mathbf{D} \) the elastic moduli tensor, respectively. \( \alpha_s \) represents the thermal expansion coefficients.
of material. $\Delta T$ is the temperature differential between current and initial temperature.

2.3. Variable stiffness method (VSM)

For aeroelastic computations, eq. (6) is rewritten as

$$M\ddot{a} + C\dot{a} + K(a)a = qf + \sum_e \int_{\Omega_e} B^T D\xi_0 d\Omega$$

(8)

where $q = \rho_\infty U_\infty^2/2$ is the freestream dynamic pressure and $f$ the nondimensional aerodynamic force vector. Therefore, following the concept of TM, eq. (8) can be solved in time domain by varying the dynamic pressure $q$ to realize flight conditions of interest for the aeroelastic and aerothermoelastic computations, e.g., the flutter onset prediction. In the present study, the freestream density ($\rho_\infty$) remains unchanged to keep the mass ratio of the calculation model constant, and the freestream velocity is varied when using the TM to perform aeroelastic computations.

To investigate the variable stiffness method (VSM) theoretically, left-multiplying the inverse of global mass matrix $M$ on both sides of the eq. (8) yields

$$\ddot{a} + M^{-1}C\dot{a} + A(a)a = qM^{-1}f + M^{-1} \sum_e \int_{\Omega_e} B^T D\xi_0 d\Omega$$

(9)

where $A(a) = M^{-1}K(a)$, and the square matrix $A(a)$ can be diagonalized as

$$A(a) = Q(a) L(a) Q(a)^{-1}$$

(10)

where the columns of the matrix $Q(a)$ are right eigenvectors of $A(a)$, and the diagonal matrix $L(a)$ is constructed by the corresponding eigenvalues.
Eq. 9 can be nondimensionalized by introducing the nondimensional time
\[ \tau = \omega_d t_{physics} \] and \( \omega_d = \omega_2(0) \)
\[
\ddot{a} + \frac{M^{-1}C}{\omega_d} \dot{a} + Q(a)\overline{L}(a)Q(a)^{-1}a = \frac{\rho_\infty U_\infty^2}{2\omega_d^2} M^{-1} f \\
+ M^{-1} \sum_e \int_{\Omega_e} B^T \overline{D} \varepsilon_0 d\Omega \\
\tag{11}
\]
where \( \overline{L}(a) = \text{diag}(\frac{\omega_1^2(a)}{\omega_d^2}, \frac{\omega_2^2(a)}{\omega_d^2}, \cdots, \frac{\omega_n^2(a)}{\omega_d^2}) \), and \( \omega_n^2(a) = \omega_n^2(0) l_n(a) \). \( \omega_n(0) \) is the \( n \)th-order circular natural frequency of the structural model at the initial configuration. \( \overline{D} = \frac{D}{\omega_d^2} \) is the nondimensional elastic moduli tensor and constant as material nonlinearity is not considered in the present study. Eq. 11 indicates that the VSM can compute the aeroelastic responses by varying \( \omega_d \) while keeping the freestream conditions such as \( T_\infty, U_\infty, \rho_\infty \) fixed. The flow chart of the VSM is illustrated in Fig. 1. Notably, the thermal effects can be neglected by setting the temperature differential \( \Delta T = 0 \).

Typically, two approaches can be used to model the damping matrix \( C \) in the structure governing equation as following

1. Viscous damping is a common form of damping, which assumes that the damping force is proportional to the speed of motion and the damping ratio of different frequencies is used to characterize the damping of the system. Therefore, Eq. 11 can be rewritten as:
\[
\ddot{a} + M^{-1} \overline{C} \dot{a} + Q(a)\overline{L}(a)Q(a)^{-1}a = \frac{\rho_\infty U_\infty^2}{2\omega_d^2} M^{-1} f \\
+ M^{-1} \sum_e \int_{\Omega_e} B^T \overline{D} \varepsilon_0 d\Omega \\
\tag{12}
\]
where the nondimensional damping matrix \( \overline{C} \) is only a function of damping ratio \( \zeta \). It is found from Eq. 12 that the VSM is equivalent to TM.
2. Classical Rayleigh damping can be another option, whereby the damping matrix $C$ is formulated as a linear combination of mass matrix $M$ and stiffness matrix $K$:

$$C = \alpha_R M + \beta_R K$$  \hspace{1cm} (13)

where $\alpha_R$, $\beta_R$ are constants and function of structural natural frequency. In this situation, Eq. 11 is formulated as:

$$\ddot{a} + \frac{\alpha_R}{\omega_d} \dot{a} + Q(a)\bar{L}(a)Q(a)^{-1} (a + \beta_R\omega_d \dot{a}) = \frac{\rho_\infty U_\infty^2}{2\omega_d^2} M^{-1} f$$

$$+ M^{-1} \sum_{e} \int_{\Omega_e} B^T D e_0 d\Omega$$

(14)

The damping coefficient matrix can be written as

$$f(\omega_d) = \frac{\alpha_R}{\omega_d} I + Q(a)\bar{L}(a)Q(a)^{-1} \beta_R \omega_d.$$  \hspace{1cm} (15)

$I$ is an identity matrix, Eq. 15 suggests that the equivalence of TM and VSM can only be established when $f(\omega_d^{TM}) = f(\omega_d^{VSM})$ satisfies.

Therefore, the VSM is not equivalent to the TM when classical Rayleigh damping is considered as $\omega_d^{TM} \neq \omega_d^{VSM}$.

Mathematically speaking, the VSM is equivalent to TM under the same similarity parameters (e.g. Mach number, Reynolds number) when solving the aeroelastic system without damping. However, some key points in the VSM for aeroelastic computation with thermal effects remain unexplained, such as how to obtain the temperature differential $\Delta T$ for the VSM and TM to reach equivalence if it is possible? Next, three representative ways to compute $\Delta T$ to consider thermal effects are presented.
Figure 1: The flow chart of VSM, whereby the temperature differential is obtained using three methods.
2.4. Temperature differential $\Delta T$ consideration

In this section, three widely used methods to obtain the temperature differential $\Delta T$ are discussed as follows\[41, 42, 10, 40\]:

(a) $\Delta T$ is a multiple of the critical temperature $\Delta T_{cr}$.

(b) $\Delta T$ is computed by steady CFD computation, where the adiabatic wall boundary condition is imposed to obtain the adiabatic wall temperature and the temperature differential can be computed through heat transfer analysis.

(c) $\Delta T$ is obtained by integrating heat flux on the solid wall.

To achieve the equivalence between the VSM and TM when using (b) to compute the temperature differential $\Delta T$, define the reference $q$ and $\omega_d$ used in the VSM and TM as $q_{ref}$ and $\omega^*_d$, whereas the $q_c$ and $\omega^*_d$ denote the current values used in the TM and VSM, respectively. The following condition needs to be satisfied to achieve the equivalence between TM and VSM, which is derived by matching the nondimensional flutter speed in the TM and VSM.

\[
q_c = \beta q_{ref}
\]

\[
\omega^*_d = \sqrt{\beta \omega^*_d}
\]

where $\beta$ is the transform coefficient. For ideal gas, it can be inferred that $\Delta T_{ref}$ and $\Delta T_c$ should obey the following equation, which can be used to compute $\Delta T$ in eq. (11)

\[
\Delta T_c = \beta \Delta T_{ref}
\]
In case (c), the heat flux is computed by imposing iso-thermal condition on the solid wall in the CFD computation, whereby the freestream temperature is set to the solid wall. Similarly to (b), the temperature differential is obtained through heat transfer analysis. To establish the equivalence of TM and VSM, the following relation needs to be satisfied as the heat flux is approximately proportional to the cube of the freestream velocity for high Mach number flow.

\[ Q_c \approx \beta^2 Q_{ref}. \]  

However, when the temperature of the iso-thermal wall is not equal to the freestream temperature for different freestream dynamic pressure, Eq. 18 is not applicable and the relationship requires further study. Consequently, the VSM is not equivalent to TM in this scenario.

3. Results and discussions

The accuracy of the in-house CFD solver used in the present study is verified in this section. All the computation in this paper is conducted on a personal computer with 8 processors (CPU: 3.6 GHz, Memory: 32GB). In the present study, the staggered algorithm, which is referred to as the Conventional Serial Staggered (CSS) procedure [13], is adopted for the aeroelastic analysis.

Unsteady flow past the oscillating NACA0012 and NACA64A010 airfoil are computed by solving Euler and RANS equations to valid the CFD solver. The parameters are summarized in table 1.
Table 1: Parameters of CT5 and CT6 cases.

<table>
<thead>
<tr>
<th>Description</th>
<th>Variable</th>
<th>Landon experiment [26]</th>
<th>Davis experiment [8]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case</td>
<td></td>
<td>CT5</td>
<td>CT6</td>
</tr>
<tr>
<td>Airfoil</td>
<td></td>
<td>NACA0012</td>
<td>NACA64A010</td>
</tr>
<tr>
<td>Mean angle of attack</td>
<td>$\alpha_m$</td>
<td>0.016°</td>
<td>0.0°</td>
</tr>
<tr>
<td>Pitching amplitude</td>
<td>$\alpha_0$</td>
<td>2.51°</td>
<td>1.01°</td>
</tr>
<tr>
<td>Mach number</td>
<td>$Ma$</td>
<td>0.755</td>
<td>0.796</td>
</tr>
<tr>
<td>Reduced frequency</td>
<td>$k_c$</td>
<td>0.0814</td>
<td>0.202</td>
</tr>
<tr>
<td>Pitching axis (%chord root)</td>
<td>$x_m$</td>
<td>25</td>
<td>24.8</td>
</tr>
<tr>
<td>Reynolds number</td>
<td>$Re$</td>
<td>$5.5 \times 10^6$</td>
<td>$12.56 \times 10^6$</td>
</tr>
</tbody>
</table>

The sinusoidal pitching motion of the airfoils is given in terms of the variation of angle of attack as a function of time,

$$\alpha(t) = \alpha_m + \alpha_0 \sin(\omega t)$$  \hspace{1cm} (19)

The reduced frequency is defined as,

$$k_c = \frac{\omega b}{U_\infty}$$  \hspace{1cm} (20)

where $b$ is the semi-chord length.

In Fig. 2, the grid dependence and time-step sensitivity study are carried out by using the RANS CFD solver with a sequence of C-type grid refinements and a set of time-steps, showing that the grid of $225 \times 65$ and the time-step of $5 \times 10^{-4}$ s are adequately accurate. To validate the CFD solver, the comparison of $C_l$ and $C_m$ is given in Figure. 3 for both CT5 and CT6 cases. The results suggest that the CFD results obtained by the in-house code agree well with the experimental data, which establishes a solid founda-
Figure 2: Studies of grid dependence and time-step sensitivity.

(a) grid dependence study using the time-step of $5 \times 10^{-4}$ s
(b) time sensitivity study using the grid of $225 \times 65$

3.1. Two dimensional aeroelastic system

In this section, two dimensional aeroelastic system is studied to validate the present VSM by using the Euler CFD solver to compute the aerodynamic forces. Fig. 4 shows the sketch of a typical two-dimensional aeroelastic system with plunge and pitch degrees of freedom (DOF). The nondimensional governing equations for the two-dimensional aeroelastic system can be described as [11, 23]

$$\ddot{\alpha} + M^{-1}K \alpha = \frac{V^2}{\pi} M^{-1} f$$  \hspace{1cm} (21)
where

\[
\mathbf{a} = [h/b \quad \alpha]^T, \quad \mathbf{f} = [-C_l \quad 2C_m]^T, \quad V_f = U_\infty/(b\omega_\alpha\sqrt{\mu}),
\]

\[
\mathbf{M} = \begin{bmatrix} 1 & x_\alpha \\ x_\alpha & r_\alpha^2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \left(\frac{\omega_h}{\omega_\alpha}\right)^2 l_1(\mathbf{a}) & 0 \\ 0 & r_\alpha^2 l_2(\mathbf{a}) \end{bmatrix}
\]

In the next section, the Isogai wing model is chosen to verify the ability and accuracy of VSM in flutter onset prediction.

3.1.1. Isogai wing model

The structural parameters of Isogai wing model (NACA64A010 airfoil) are [20]

\[a = -2.0, \quad x_\alpha = 1.8, \quad r_\alpha^2 = 3.48, \quad \omega_h/\omega_\alpha = 1.0, \quad \mu = 60, \quad l_1(\mathbf{a}) = 1, \quad l_2(\mathbf{a}) = 1\]
In this study, the coefficients about the quarter-chord at a reduced frequency per period of forced oscillation are taken. The airfoil is forced in pitch freestream Mach number far-field boundary located at a radius of 20 chords. To validate the points, among which 160 are on the airfoil surface and 32 are on the 

The NACA 64A010 airfoil domain is discretized by using characteristic time for the aerodynamic part, unsteady Euler method with unstructured dynamic meshes [28]. The pitching motion of the airfoil is described by the following equation:

$$\dot{x} = \omega^2 x$$

Because both aerodynamic and structural response sustains. The temporal evolution of the response is given in Fig. 5, can been seen in the figure. The typical S-shape flutter onset boundary is.

![Sketch of the airfoil section of two degrees of freedom.](image)

To enable the system transform into fully developed state rapidly, initially, the airfoil pitches around the elastic axis sinusoidally for two periods with frequency \( \omega_a \), after which the evolution of the system is driven by self-induced loads. The Euler CFD solver is used with an O type grid 165 × 59. Structural displacements are computed by the Adams-Bashforth-Moulton scheme [46]. In the VSM, the pitching natural frequency is chosen as the adjustable frequency \( \omega_d \).

The flutter onset is determined by the lowest \( V_f \), for which the aeroelastic response sustains. The temporal evolution of the response is given in Fig. 5, showing that the flutter is achieved at \( V_f = 0.82 \). The nondimensional flutter speed and frequency are plotted as a function of freestream Mach number in Fig. 6. A good match between the results of time domain (TM) and VSM can been seen in the figure. The typical S-shape flutter onset boundary is
Figure 5: Typical temporal evolutions of the unstable mode computed by the VSM at $Ma = 0.8$. 
observed and consistent with previous research [47, 17].

It is worth noting that the VSM predicts the flutter onset by varying the pitching natural frequency $\omega_d$ while keeping the time step $\Delta \tau$ fixed. Fig. 7 shows the flutter onset responses at $Ma = 0.8$ and nondimensional velocity $V_f = 0.82$. In the figure, the result of TM is computed by using $\Delta t_{physics} = 1 \times 10^{-4} s$, which corresponds to the nondimensional time step $\Delta \tau = \omega_d^{ref} \Delta t_{physics}$. The theoretical investigation described in section 2.3 shows that the VSM is mathematically equivalent to the TM for aeroelastic computations provided the both methods adopt the same nondimensional parameters (e.g. $Ma$ for inviscid flow), nondimensional velocity ($V_f$) and time step ($\Delta \tau$). To further confirm the theoretical investigation, the VSM is compared with the TM in the Fig. 7, where a good agreement between the results of the TM and VSM can been seen. As expected, notable discrepancy is observed between the TM and VSM($dt$), where the VSM($dt$) denotes the VSM results obtained by using the same $\Delta t_{physics}$ as its TM counterpart.
Figure 7: Aeroelastic responses an aerodynamic coefficients computed by the TM and VSM at $Ma = 0.8, V_f = 0.82$. 
3.1.2. Limit Cycle Oscillation Computation

In this section, the VSM is further assessed for nonlinear limit cycle oscillation (LCO) computation. The structure of advanced missile fin is usually featured with pitch-plunge free-play nonlinear stiffness as shown in Fig. 8. In the study, a NACA0012 airfoil with pitch and plunge DOF [23] is adopted, whereby both free-play and aerodynamic nonlinearities are considered. A C-type grid with $113 \times 33$ nodes is used and Euler equations are solved to get the aerodynamic force. The Mach number and initial angle of attack are, respectively, $[Ma, \alpha] = [0.8, 0^\circ]$. As pointed by Kim and Lee [23], the pitch free-play plays a dominant role in this aeroelastic system, therefore, only pitch free-play is considered herein. The structural parameters of the aeroelastic system with free-play nonlinearity are given by

$$a = -0.25, \quad x_\alpha = 0.25, \quad r_\alpha^2 = 0.395641, \quad \omega_h/\omega_\alpha = 0.708, \quad \mu = 36.15, \quad \alpha_f = 0.5^\circ,$$

$$l_1(a) = 1, \quad l_2(a) = \begin{cases} 1 - \frac{\alpha_f}{\alpha} & \alpha \geq \alpha_f, \\ 0 & -\alpha_f < \alpha < \alpha_f, \\ 1 + \frac{\alpha_f}{\alpha} & \alpha \leq -\alpha_f, \end{cases}$$

where $\alpha_f$ is the initial free-play magnitude of the pitch motion.

The reference frequency is chosen as $\omega_{ref} = 100 \text{ rad/s}$ in this case, and the nondimensional time step used is $\Delta \tau = 0.05$. The fourth order Runge-Kutta method is adopted to solve the structural equation with the nonlinear free-play. As shown in Fig. 9, the pitch amplitude and frequency as a function of nondimensional flutter velocity is plotted. The results show that the VSM predictions are in good agreement with the TM results. Included are
also the solution by Kim et al [23] and He et al [17], showing a similar trend, although the VSM over-predicts the pitching amplitude slightly. Fig. 10 further confirms that VSM can predict the nonlinear LCO with both structural and aerodynamic nonlinearities.

### 3.2. Three dimensional aeroelastic system

In this section, the VSM is further assessed for three dimensional fixed-wing LCO computation. The test case is considered as the Goland$^+$ wing with a tip store, where $^+$ denotes the heavy version of the wing. This model is reported to exhibit aeroelastic instabilities driven by oscillation of shockwave [3, 2] and has been investigated extensively. The wing is of rectangular shape with a constant airfoil profile, which is defined as [3]

$$
\frac{z}{c} = \pm 2\tau_0(1 - \frac{x}{c})(\frac{x}{c}),
$$

where $\tau_0 = 0.04$ is the thickness-to-chord-ratio, and $c = 1.8288$ m the chord length. The finite element model (FEM) is built by following the details
(a) Pitching amplitude versus nondimensional flutter velocity

(b) Pitching amplitude versus nondimensional flutter frequency

Figure 9: LCO solution by the TM and VSM, and the results taken from Kim et al [23] and He et al [17].

(a) time domain

(b) frequency domain

Figure 10: Pitching amplitude of LCO at $V_f = 0.283$: (a) is drawn every five data points for clarity.
provided in Ref. [3].

As shown in Fig. 11, an O-O type grid of $161 \times 61 \times 81$ is used for the inviscid computation, whereby the tip store is not reflected as the LCO is mainly driven by aerodynamic nonlinearity. A linear structure system is adopted and decoupled by retaining the first four modes as depicted in Fig. 12.

The freestream condition is $[Ma, \alpha] = [0.92, 0^\circ]$ and inviscid CFD solver is used to solve the fluid system. The second order natural frequency (torsional mode) is chosen as the reference frequency $\omega_d^{ref}$ in the VSM. Similar to the Isogai wing model, the fourth order Runge-Kutta method is adopted to integrate the structure system. Fig. 13 shows the comparison between LCO solutions predicted by VSM and TM, whereby $[\xi_1 \ \xi_2 \ \xi_3 \ \xi_4]^T$ denotes the generalized structural displacement. The results confirm the ability of the VSM for nonlinear LCO computation of three-dimensional fixed wing. In section 3.4, the VSM is further demonstrated for situation where the thermal effect is considered.

3.3. The effect of damping

In this section, the VSM and TM are verified for solving aeroelastic system with damping, which is formulated by the two approaches presented in the section 2.3. Fig. 14 shows the schematic diagram of the set up for the numerical verification. The material and geometry parameters are given by

$$
\begin{align*}
E_s &= 70 \text{ GPa}, \quad \nu = 0.3, \quad \alpha_s = 2.4 \times 10^{-5} / \text{K}, \\
\rho_s &= 2800 \text{ kg/m}^3, \quad a_s / h_s = 500.
\end{align*}
$$

The simplified verification process is as follows:
Figure 11: Inviscid O-O type grid of Goland wing.
Figure 12: The first four elastic mode shapes of Goland+ wing.
Figure 13: Aeroelastic responses at $U_\infty/(\omega_d c) = 4.4626$: (a) is drawn every five data points for clarity.
1. A uniform pressure load $\Delta P_{TM} = P_1 - P_2 = 2 \text{ Pa}$ is imposed on the top surface of the plate and the responses with/without damping are calculated by TM.

2. The reference pressure load used for VSM is $\Delta P_{VSM} = \beta \Delta P_{TM}$, $\beta = 4$, therefore the structure stiffness is also quadrupled to match the nondimensional velocity according to Eq. 16.

The structural response is compared between TM and VSM at $x/a = 0.5$, where the aeroelastic system rises to the peak response. As shown in Fig. 15, the results of TM and VSM match well when using Eq. 12 to define the damping matrix, which is consistent with the theoretical derivation in section 2.3. However, the VSM is not equivalent to its TM counterpart when the classical Rayleigh damping is used to define the damping matrix, which is confirmed by the results shown in Fig. 16, where notable discrepancy between the TM and VSM is observed in the response with
damping. For $[\alpha_R, \beta_R] = [3, 0.0001]$, the damping coefficient used in TM is $rac{\alpha_R}{\omega_d^2} I + Q(a)L(a)Q(a)^{-1}\beta_R\omega_d^T$, whereas the damping coefficient used in VSM is $rac{\alpha_R}{\omega_d^{VSM}} I + Q(a)L(a)Q(a)^{-1}\beta_R(\omega_d^{VSM})$ as $\omega_d^{VSM} = \sqrt{\beta}\omega_d^TM$, $\beta = 4$ is used to match the nondimensional flutter speed. As $\alpha_R$ is 4 orders of magnitude larger than $\beta_R$, the damping in VSM is smaller than that in TM. Therefore, VSM produces higher amplitudes than its TM counterpart.

3.4. Aeroelastic system with thermal effects

In the present study, the aeroelastic system is considered as the same model described in Section 3.3 which subject to uniform thermal load with freestream condition $[Ma, \alpha] = [1.2, 0^\circ]$ and mass ratio $\mu = \frac{\rho_a}{\rho_s h_s} = 0.1$ [10, 15]. Considering that the thermal effect is the focus of this section, the damping is neglected for all computations. The nondimensional dynamic pressure and the stiffness are, respectively, defined as

$$\lambda = 2qa_s^3/D, \quad D = E_s h_s^3/12(1 - \nu^2).$$
The validity of the VSM is first verified by predicting the flutter onset of the panel without the thermal effect. As shown in Fig. 17, VSM agrees reasonably well with the literature [10, 15]. The VSM for aeroelastic computation subject to uniform thermal load is demonstrated as follows.

The temperature differential $\Delta T$ is considered as a multiple of the critical temperature $\Delta T_{cr} = \frac{\pi^2 h_s^2}{(12 (1 + \nu) \alpha_s a_s^2)}$. A set of four representative temperature differential $\Delta T/\Delta T_{cr} = 1, 2.5, 5, 10$ is chosen to demonstrate the VSM for aeroelastic computation, the selected nondimensional dynamic pressure in TM is $\lambda^c = 40$ and the corresponding reference value is $\lambda^{ref} = 10$ in VSM.

As shown in Fig. 18 and 19, the structural responses experience a distinct increase in amplitude and frequency as the temperature differential $\Delta T$ increases, which eventually leads to the structure buckling. The results confirm that the VSM is adequate to capture the nonlinear dynamics of the
Figure 17: Validation of the VSM for flutter onset prediction. \( w/h \) is the displacement normalized by the panel thickness \( h_s \) at \( x/a_s = 0.75 \), and \( K_f = \omega/\omega_0 \) is nondimensional frequency, where \( \omega_0 = \pi^2 \sqrt{D/(\rho_s h_s a_s^4)} \).

As mentioned earlier, the temperature differential \( \Delta T \) can be obtained by steady CFD computation. It worth noting that the condition \( \Delta T_c = \beta \Delta T_{ref} \) needs to be satisfied when comparing TM and VSM for the aeroelastic computation with thermal effect.

As mentioned earlier, the equivalence between the VSM and TM can only be achieved approximately when \( \Delta T \) is computed by heat flux at high Mach numbers and the solid wall temperature \( T_w \) is equal to freestream temperature \( T_\infty \). This is further elucidated by comparing \( Q_c \) with \( Q_{ref} \) in the Fig. 20. The results are computed by the Menter \( k – \omega \) SST turbulence model at \( Ma = 1.2, Re = 16.78 \times 10^6, \alpha = 0^\circ \) with the first cell height at wall \( \delta y = 10^{-6} \text{ m} \). As shown in the figure, notable discrepancy can be observed between \( Q_c \) and \( \beta^{3/2}Q_{ref} \), whereas \( P_c \) matches \( \beta P_{ref} \) quite well. Therefore,
Figure 18: Structural responses for different temperature differentials. Figures are plotted every forty data points for clarity.
Figure 19: Phase portraits for different temperature differentials.
the VSM is only approximately equivalent to its TM counterpart in this scenario. Another alternatively treatment is to set the solid wall temperature to $T_w = 1.5187$ K (the temperature is obtained at $\beta = 1, \lambda = 10$ in this study) under different dynamic pressure, the corresponding comparisons are illustrated in Fig. 21. It can be seen that $P_c$ still agrees quite well with $\beta P_{ref}$, but the differences between $Q_c$ and $\beta^{3/2} Q_{ref}$ are much larger than that shown in Fig. 20.

3.5. VSM for ARMA/CSM model construction

In this section, the versatility of the VSM is further demonstrated by a rudimentary ROM construction. For this purpose, the autoregressive moving average (ARMA) model is adopted and can be written as [6]:

$$y(k) = \sum_{i=1}^{na} [A_i] y(k-i) + \sum_{i=0}^{nb-1} [B_i] a(k-i)$$  \hspace{1cm} (24)
where $y$ and $a$ are the system outputs and inputs vector. $[A_i]$ and $[B_i]$ are the matrices of constant coefficients. $na$ and $nb$ denote the input and output delay, respectively. Eq. 24 suggests that the output at time step $k$ is the linear combination of past inputs and outputs. The ARMA model requires a training signal covering the parameter of interest (i.e. frequency) to identify the matrices and delays in Eq. 24. In the present study, the “3211” multistep signal is imposed on the velocity of the solid wall [6] for training process. In the VSM framework, the resultant ARMA/CSM model can be formulated by integrating the ARMA model into the structural equation Eq. 11. The ARMA/CSM model can be then used to compute structural responses at different nondimensional flutter velocity by varying the frequency $\omega_d$. The computational cost required by the ARMA/CSM model and VSM is given in table 2. As expected, the expensive part of the ARMA/CSM model construction is the training process. However, the overhead spent for training

Figure 21: Pressure and heat flux versus transform coefficient $\beta$ for $T_w = 1.5187 \, K$ (computed at $\beta = 1, \lambda = 10$).
is only once. With the ARMA/CSM model constructed, it can be used to compute the flutter onset of the aeroelastic system at a fraction of the cost of the FOM. The Isogai wing model used in section 3.1.1 is chosen to verify the constructed ARMA/CSM model. Fig. 22 shows the comparison of aeroelastic responses at flutter onset between TM, VSM and ARMA/CSM model. As expected, the aeroelastic ROM predictions are in good agreement with the VSM and TM results. A slight discrepancy is observed in the plunge response, where the ARMA/CSM model overpredicts its full order model counterpart or VSM by about 2%.

Table 2: Computational cost of the ARMA/CSM model and VSM.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Details</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>VSM</td>
<td>1 aeroelastic simulation (6000 time steps)</td>
<td>10 h</td>
</tr>
<tr>
<td>ARMA/CSM model</td>
<td>1 training case (2000 time steps in total)</td>
<td>2.22 h</td>
</tr>
<tr>
<td></td>
<td>+ ARMA/CSM model computation (6000 time steps)</td>
<td>+ 3 s</td>
</tr>
</tbody>
</table>

4. Conclusions

The variable stiffness method (VSM) is revisited theoretically and numerically for nonlinear aeroelastic computation with/without thermal effects. The VSM predicts system responses by varying the reference frequency in structural equation while keeping the freestream conditions (i.e. $\rho_\infty$, $U_\infty$) fixed. The method is first validated by a two-dimensional aeroelastic system of the Isogai wing section for flutter onset prediction. This is followed by limit cycle oscillation (LCO) computation of the Goland+ wing to further confirm the ability of the method in more practical applications. To high-
Figure 22: Aeroelastic responses at $Ma = 0.8, V_f = 0.82$. 
light the versatility of the VSM, aeroelastic computation is conducted on a simply-supported plate subject to thermal effects in the supersonic regime. However, both theoretical analysis and numerical simulation have shown that the condition for the equivalence between the VSM and TM can not be satisfied for aeroelastic computation when Rayleigh damping is considered or the heat flux is computed by means of CFD for heat transfer analysis. Following the same concept of the VSM, a rudimentary ROM is built by the ARMA model and demonstrated to be accurate and efficient in flutter onset prediction. The results obtained in the present study demonstrates the potential of the VSM for aeroelastic and aerothermoelastic computation (i.e. flutter onset, LCO and buckling).

Acknowledgments

The first author gratefully acknowledges the support from the College of Astronautics, Northwestern Polytechnical University. This work is supported by the National Natural Science Foundation of China (Grant No. 11602296).

References


